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Espresso

A sip of filtered research

CAMBRIDGE Mathematics

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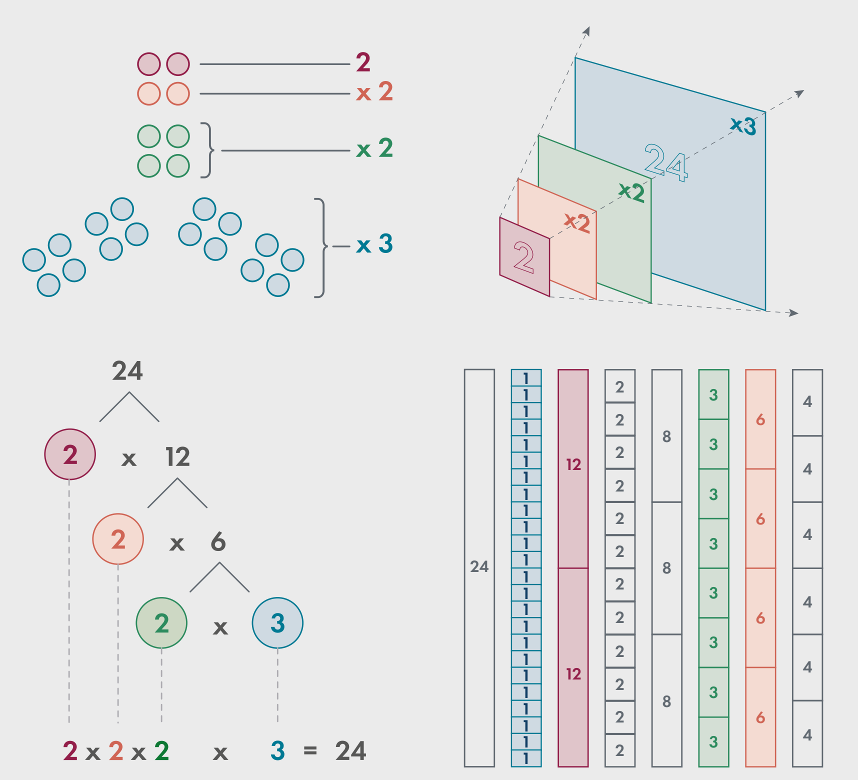
Talking point

What does research suggest about the teaching around factors, multiples and prime numbers?

In summary

- Decomposing numbers to investigate their multiplicative structure can support a flexible approach to problem solving and should come before the introduction of rules or procedures
- Activities in which students: sort objects into regular arrays of width two; explore which numbers can be split into two equal groups; and also into equal groups of two, all support conceptual understanding that even numbers are divisible by two
- Students need to link doubling to multiplying in order to appreciate that it will always result in an even number
- Visualising building numbers by scaling or growing, rather than by repeated addition, helps support multiplicative reasoning and generalising
- Working with characteristics of primes can help avoid misconceptions about their size and prevalence as factors of other numbers
- Practising seeing prime factors both individually or in combinations, can help support flexible reasoning about the divisibility of the whole number
- Making links between different methods of finding the lowest common multiple of two numbers can support conceptual understanding; Venn diagrams are suggested as useful ways to visualise common prime factors of two numbers

Recognising 24



1 Developing an awareness of the underlying multiplicative structure of whole numbers can support a flexible approach to problem solving.[3] Creating and considering equivalent representations of the same number (see infographic for examples) and the related concepts of factors, multiples and divisors, can reveal general properties about families of numbers such as odd, even, composite and prime.[4] Research is scarce on school students' experience of using primes to build numbers, but some exists on teachers' learning of and teaching around these concepts.[5,6,7]

Implications:

Decomposing whole numbers to investigate their structure can support a flexible approach to problem solving

Considering the multiplicative structure of numbers can highlight generalisations about their properties

> "That's part of what makes primes so interesting: not only is the number line studded with primes all the way up to infinity, but that whole number line can be produced using nothing but primes"
>
> Templeton, 2020[1]

> "The operation of addition or subtraction is the same as extending or cutting off lines; the product of two numbers a and b is the same as the geometric construction of a rectangle having adjacent sides a and b"
>
> Mazur, 2016[2 (p143)]

2 Odd and even numbers are often some of children's first experiences of seeing the multiplicative structure of numbers. It is suggested that activities in which students physically sort groups of objects into regular arrays of width two can support a recognition of the way even numbers form a rectangle and odd numbers form an L-shape, and this supports pattern-spotting and generalisation.[8] Providing the opportunity to explore which numbers are made of pairs, can be split into two equal groups, and can be split into equal groups of two is more helpful than a reliance on identifying the last digit of a number as a characterisation of the "two-times-table".[9] Students do not necessarily connect "doubling" with multiplying by two,[10] or recognise that when operating on whole numbers, doubling will always produce an even number, and so need support to do so.[11]

Implications:

Activities in which students sort objects into twos can helpfully lead to the visualisation and then generalisation that even numbers can form a rectangle and odd numbers an L-shape

Exploring which numbers can be split into two equal groups and also into equal groups of two supports conceptual understanding that even numbers are divisible by two

Learning a list of digit endings as a way of identifying odd and even numbers does not support conceptual understanding of whether or not the number has a factor of two

Doubling needs to be explicitly linked to multiplying for students to appreciate when it will always result in an even number

3 A variety of representations and contexts are suggested for exploring multiples (numbers that divide exactly by another without a remainder), with the goal being for students to identify patterns and be able to generalise characteristics.[12] Conceptual understanding of divisibility should be established before introducing divisibility rules and procedures.[6] Vocabulary needs careful attention, as reasoning with and about multiplicative number sentences can lead to words such as "multiple" (an object) and "multiply" (a process) being mistakenly used interchangeably.[5] Development of multiplicative reasoning can be blocked if multiples are visualised via repeated addition rather than scaling or growing and this creates difficulties in coming to understand a number's structure as a product of its prime factors.[13]

Implications:

Using a range of representations and contexts when exploring multiples supports students' pattern-spotting and generalising

Conceptual understanding of divisibility should be established before rules or procedures are introduced

Careful consideration of the specific vocabulary being used (odd, even, factor and multiple) is recommended

Visualising composing numbers multiplicatively (by scaling or growing) rather than additively helps support multiplicative reasoning and generalising

4 A tendency to define primes by what they are not[7] can result in misconceptions that prime numbers are small (under 100) and that every large number, if composite (having more than two factors), is divisible by a small prime number. Students have a tendency to calculate the product of prime factors and then divide the answer to check for divisibility, rather than reasoning flexibly.[6]

Implications:

Working with characteristics of primes (rather than what they are not) can help avoid misconceptions about their size and prevalence as factors of other numbers

Practising seeing prime factors both individually and in combinations can help support flexible reasoning about the divisibility of the whole number

5 There are three common approaches to finding the lowest common multiple: set intersection (finding a number that appears in a list of multiples for each); creating a multiple and then dividing (checking for divisibility by the second number through an ordered list of multiples of the first); and prime factorisation (finding the minimal product of prime powers that contain both their factorisations) – but students often seem unable to recognise the equivalence of these methods.[13] A useful suggested approach to finding the highest common factor of two numbers is using a Venn diagram representation, where each circle (set) contains the prime factors of its respective number, with any shared prime factors placed in the overlap. These shared prime factors can then be multiplied to find the highest common factor.[4]

Implications:

Making links between different methods of finding the lowest common multiple of two numbers can support conceptual understanding, relating to factors and multiples, and flexible problem-solving

A Venn diagram can be used to visualise common prime factors of two numbers, which can then be multiplied together to find the highest common factor

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