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Example Research Summary Package

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Sections in this document

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Research Summary: *Early angle conception and measurement*

Author: Rachael Horsman

Research landscape

Angles combine ideas of both geometry and measurement. Multiple definitions exist, with angles being considered as (a) a geometric shape, (b) a measure, or (c) a dynamic rotation (Henderson and Taimina, 2005). It seems sensible, if not essential, to support learners in integrating both the static and the dynamic (described below) in order to understand angle measure (Clements & Battista, 1989, 1990; Mitchelmore & White, 2000).

In the work of Mitchelmore and White (2000, p.214-216) three stages of abstraction are identified:

- situated angle concepts where students identify, characterise and operate on angles;
- contextual angle concepts where students establish properties of angles and use these to solve problems; and
- abstract angle concepts where students formulate and use definitions, give informal arguments and hierarchies to previously discovered properties and finally develop deductive arguments.

This Research Summary concentrates on the first of these stages.

When exploring angles a variety of situations and contexts will be investigated. Initially the concept starts with children's own dynamic perspective: enacting turns for themselves (Gates & Griffin, 1988), watches and wheels (Magina, 1994); alongside their intuitive use of parallel and perpendicular when they first start trying to build with blocks and similarly when they align tiles (Sarama and Clements, 2009). Completing tangram-type puzzles gives opportunities to compare the size and hence congruence of angles and angle measure. Drawing these activities and discussions together, bringing them into geometry lessons and making connections to the angle measure, enhances understanding (Sarama and Clements, 2009).

It is advised that three distinct settings are investigated, independently to begin with (Mitchelmore and White, 2000):

Rotations

It is possible to categorise rotations as either limited or unlimited about an interior point: e.g. a key rotating in a lock (limited), the hands of a clock (unlimited). Rotations can lack a clearly identified centre and rotating line hence the association to angle may not always be made by learners.

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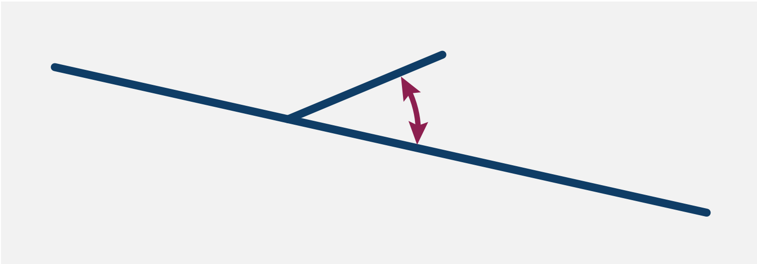
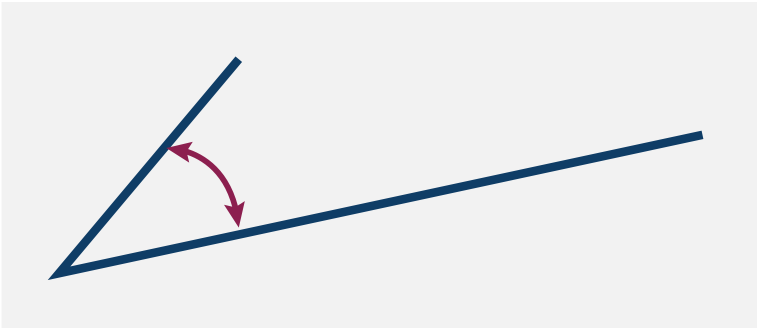
Angles (hinges)

Hinges are the prototypical angle. Mitchelmore (1998, p.269) categorises these into three distinct types:

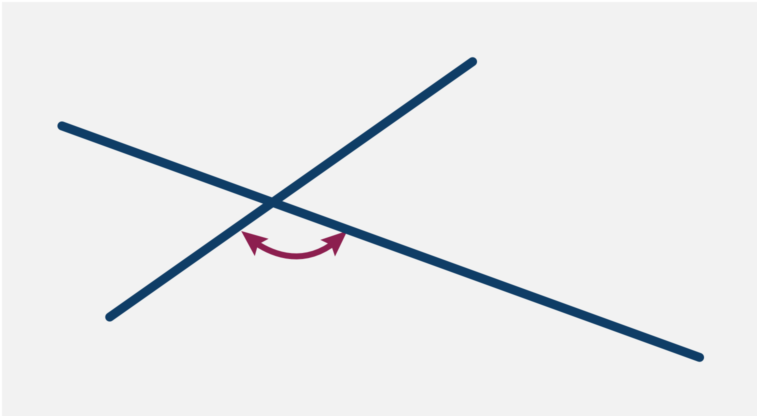
- I-Hinges – a single, linear object hinged about one end rotating between well-defined limits; e.g. a door



- V-Hinges – two linear objects hinged about a common end point; e.g. a book cover

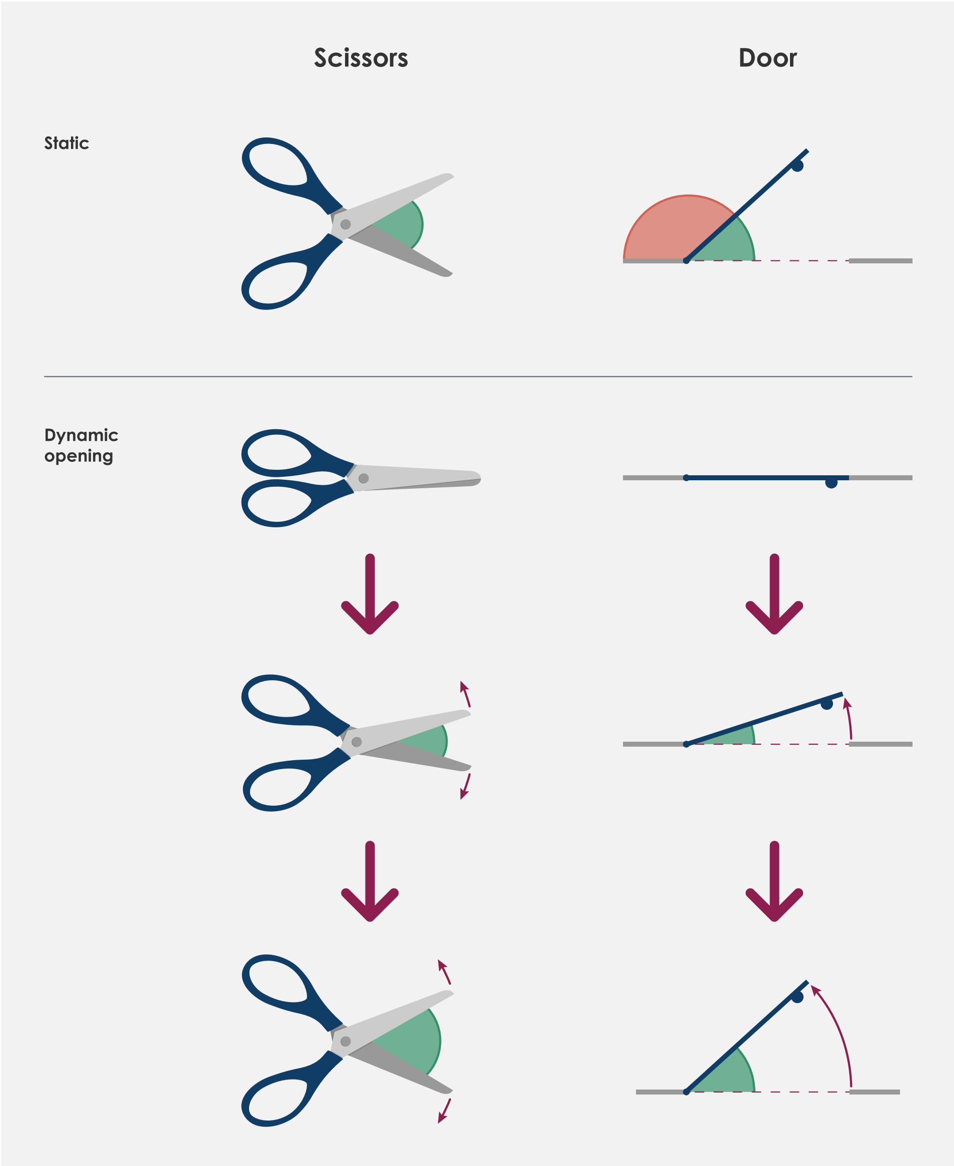


- X-Hinges – two linear objects hinged about a common interior point; e.g. scissors



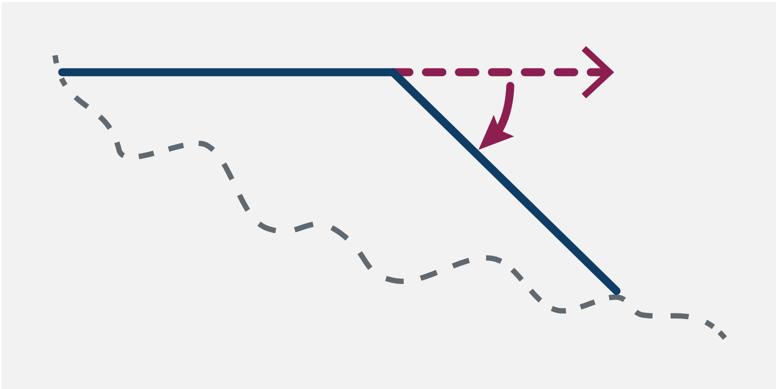
Importantly, these can be viewed as static or dynamic (Magina, 1994, p.45) and present a good opportunity to invoke the principle of variation in terms of direction and orientation of an angle to be measured.

The definition of a dynamic angle considers it as a measure of quantity of rotation needed to bring one of its sides from its own position to that of the other side (Heath, 1956, p. 179 in Magina, 1994). Mitchelmore (1998) models this by considering a pair of scissors. The dynamic model would show the scissors opening; a static model the result of that opening. In many cases the two models are not in conflict. However, at times more careful thought is needed – such as with an open door. The dynamic model may consider the angle through which the door is opened whereas the more natural static one would be to measure the angle between the given position and the fully open position (resulting in two different angles). Hence, a correct, unambiguous and efficient labeling system is needed.



Bends (angular)

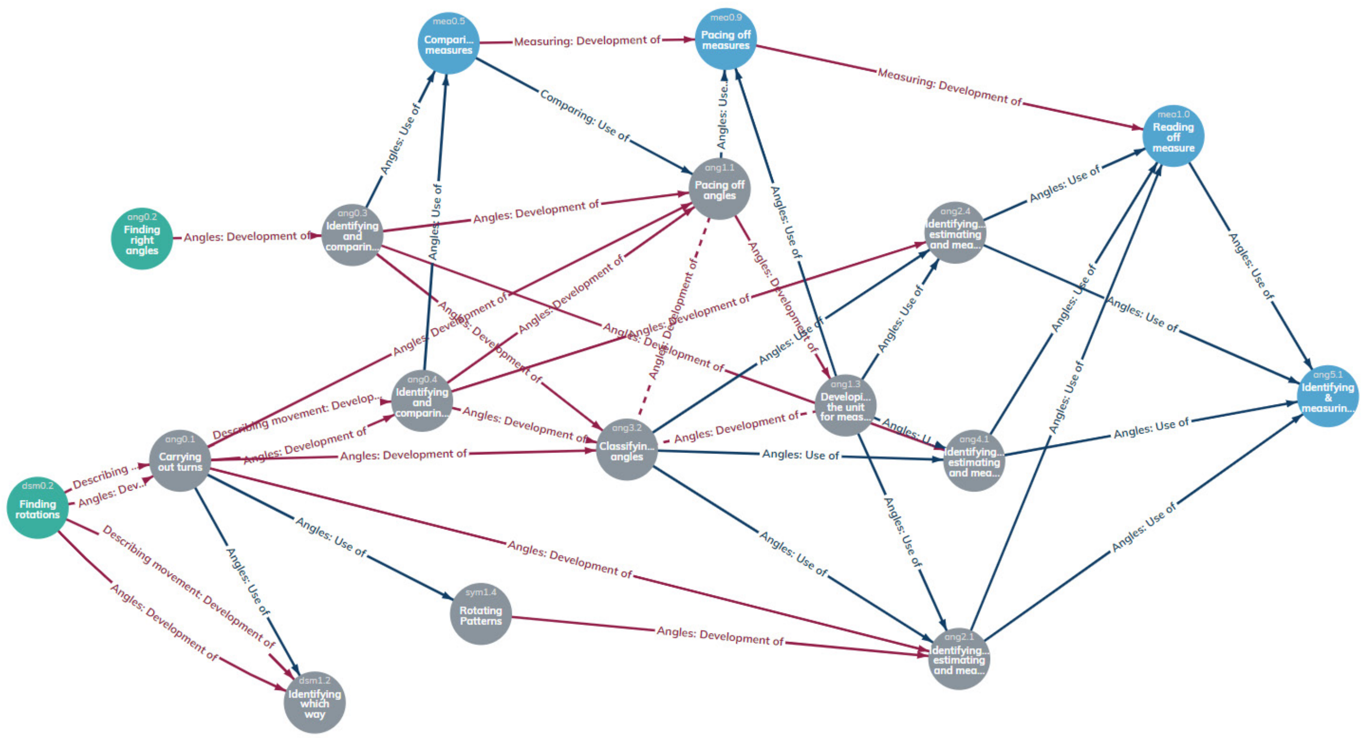
Bends are defined as two line segments with common endpoints. Moving forward along one line, a turn is needed to continue the journey along the second line: e.g. external angles of a polygon, angles of inclination, turtle geometry. In contrast, curves and arcs will be consider in more detail in a later research summary, although they may be compared and contrasted to angular bends within the waypoints discussed below.



Providing opportunities for students to explore each of these contexts is critical for the development of their understanding of geometry (Browning et al., 2007).

After their explorations of these three independent settings students are encouraged to become aware of their shared characteristics, collecting their experiences together. As Skemp (1986) describes, this abstracting (classifying according to these similarities) results in an abstraction: in this case the concept of angle.

Framework map for Early angle conception and measurement



Key: Exploratory waypoint Landmark waypoint Waypoint

Development of concept, skill, procedure Use of concept, skill, procedure

Implications for the Framework

As in the other geometry themes, specifically those dealing with measurement, the initial waypoints in this saved search begin with an exploration of the concept. Pupils identify objects that rotate (dsm0.2) linking in with the *Describing Movement* theme. Pupils identify angles, in 2D and 3D contexts in both static and dynamic situations (ang0.3) and in instances where forward motion is combined with bends or slopes(ang0.4). Importantly in all these waypoints static and dynamic examples will be considered, including some examples of embodied maths such as following lines on a playground.

In each of these settings, angles are compared directly – superposing angles/representations of turn (ang0.1, ang0.3, ang0.4); in addition some attention is paid to representing an angle between two planes using two line in 2D (ang0.3). Clear representation and conventional labeling of angles is also discussed in these waypoints and each also connects to a waypoint concerning the conventions of labeling line segments (outside this saved search).

A unit of measurement is developed through the initial use of right angles as a unit (ang0.1, ang0.2, ang3.2). Then (as with other measuring themes) there follows the development of interim arbitrary units whilst pacing off angles until finally the unit of degree is introduced as the 'standard' unit (ang1.1). At the same time benchmarks are refined, such as 45˚, linking to the number waypoint *Seeing Double* (ast1.3) (outside this saved search). This follows the development of early measurement mirrored in the areas of length, area and volume. Further information can be found in the Research Summary Developing a sense of measure.

Each of the settings is revisited in order to identify, estimate and measure the angles (ang2.1, ang4.1, ang2.4). Finally the settings are drawn together (ang5.1) resulting in pupils identifying their similarities, developing a rich and secure understanding of the concept of an angle and its measurement.

This theme also overlaps significantly with the themes of: *2D Shapes*; *Loci and Construction*; *Describing Position*; *Describing Movement*; *Isometric Transformations*; *Congruence*; and *Symmetry*.

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Example activity types from the Swan Framework, as applied to student actions

	Student Action	"Sample classroom activities." (Swan, n.d., p. 1)
Procedural Fluency	Performing	• "Rehearsing well-defined procedures through exercises and études. • Systematically using and memorising terms and notations."
Conceptual Understanding	Classifying	• "Observing and manipulating mental objects. • Identifying and describing attributes and sorting objects accordingly. • Creating and identifying examples and non-examples. • Creating and testing definitions."
	Representing	• "Interpreting a range of representations including diagrams, graphs, and formulae. • Translating between representations and studying the co-variation between representations."
	Analysing	• "Studying and modifying mathematical situations. • Exploring relationships between variables. • Comparing and making connections between mathematical structures."
	Arguing	• "Making and testing mathematical conjectures and procedures. • Identifying examples that support or refute a conjecture. • Creating arguments that explain why conjectures and procedures may or may not be valid."
	Estimating	(this student action is not part of Swan's original framework)
Problem Solving	Modeling	• "Making suitable assumptions to simplify a situation. • Representing a situation mathematically. • Identifying significant variables in situations. • Generating relationships between variables. • Identifying accessible questions that may be tackled within a situation."
	Solving	• "Planning an approach to a problem. • Selecting and applying appropriate mathematical concepts and procedures. • Selecting and using mathematical tools, including technology. • Carrying out a plan, monitoring progress and changing direction, where necessary. • Making generalisations based on the results."
	Conference participants	• "Interpreting, adopting and continuing a strategy. • Comparing alternative strategies, identifying relative strengths, weaknesses and domains of application. • Reflecting on solutions: examining for reasonableness within the context. • Reflecting on strategies and arguments: where might they have been improved? • Interpreting and testing mathematical models: Are they adequate? What are their limits? • Making connections with previously encountered problems."

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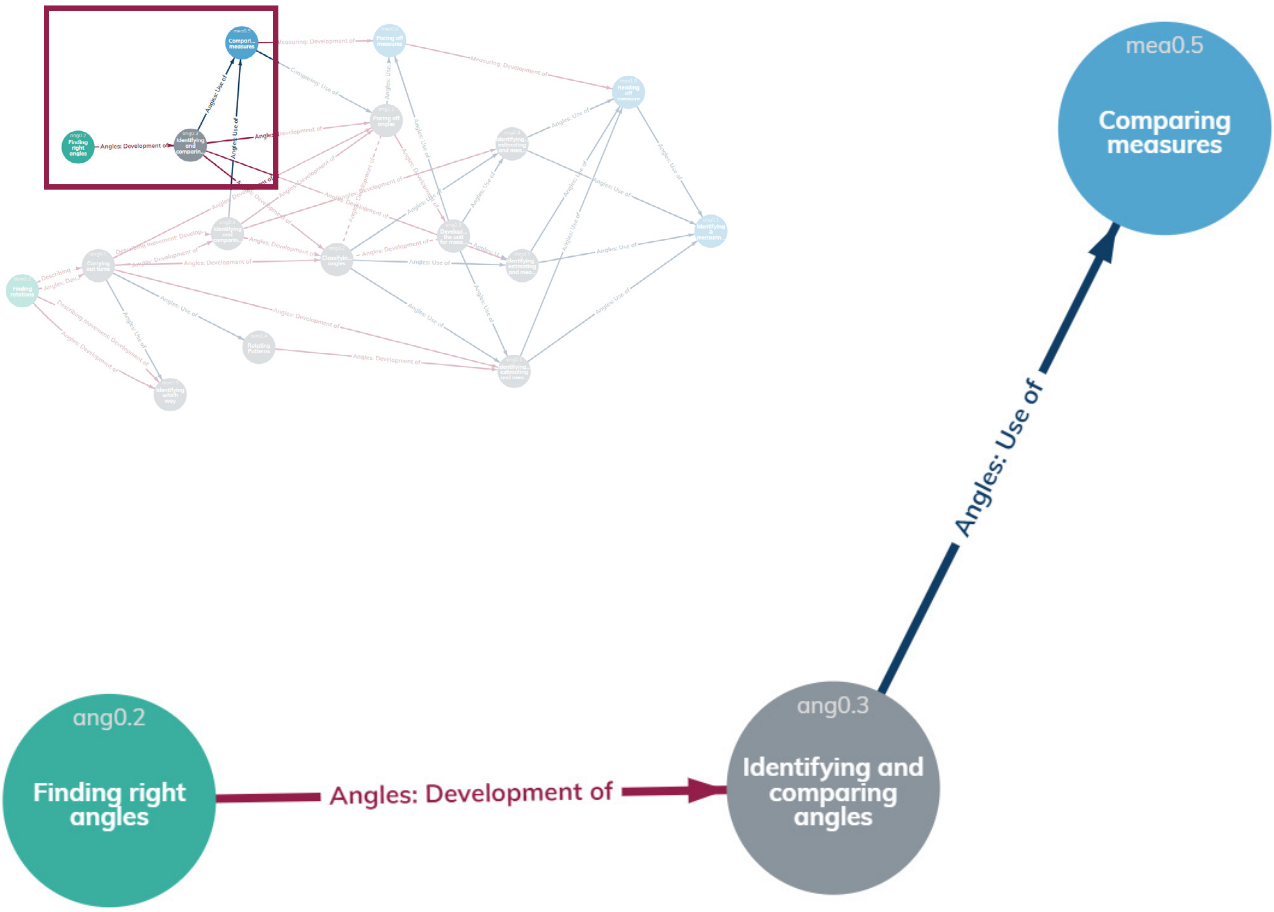
Three waypoint content examples: exploratory waypoint, waypoint and landmark waypoint

Figure 1: Inset map showing the waypoints whose details are listed below



Key: Exploratory waypoint · Landmark waypoint · Waypoint
Development of concept, skill, procedure · Use of concept, skill, procedure

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Example waypoint details

(glossary links under development; not shown)

ang0.2 Finding right angles

Type

Exploratory waypoint.

What

Identify right angles, and multiples of, in 2D and 3D using practical equipment.

Why

To develop benchmarks for angle measurement;

to recognise the importance and high frequency of occurrence of right angles and their multiples;

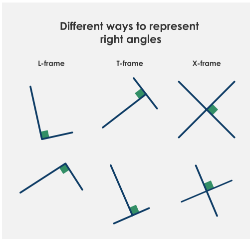
to begin to build understanding of the properties used to define or classify shapes.

Student Actions

Performing
- Identify right angles in objects and images.

Classifying
- Sort objects, images and geometric patterns into groups of those with and without right angles.

Representing
- Represent right angles in a variety of ways, for example: using finger and thumb; corner cards; traced angles; and frames, including L's, T's and X's.



Analysing
- Explore how right angles fit together and the relationship to a straight line and around a point.

Leads out to

ang0.3	by theme Angles
2dc1.1	by theme Angles
3dc1.3	by theme Angles
2dc1.1	by theme Disembedding and embedding
2ds2.1	by theme 2D
3ds4.1	by theme 3D

Glossary

This node references 2D, 3D, angle, corner, geometrical, line, measure, multiple, pattern, point, property, right angle, shape and straight.

Research

This node is linked to

- Clements, D., & Sarama, J. (2009). Composition and Decomposition of Shapes. In *Learning and Teaching Early Math: The learning trajectories approach* (pp. 149–162). New York: Routledge.
- Clements, D. H., & Sarama, J. (2000). Young Children's Ideas About Geometric Shapes. *Teaching Children Mathematics,* 6(8), 482–488.
- Mitchelmore, M. (1992). Children's Concepts of Perpendiculars. *Proceedings of the 16th International Conference on the Psychology of Mathematics Education*, 2, 120–127. Durham, New Hampshire.

ang0.3 Identifying and comparing angles

Type

Waypoint.

What

Identify and compare angles in 2D and 3D.

Compare angles: predict, justify and confirm practically whether one is larger, smaller or equal to another.

Why

To build the concept of an angle as a measurement between two lines that meet (or cross) at a point or two planes that meet along a line.

To introduce the beginning of number sentences, through the informal use of <, >, and =.

Student Actions

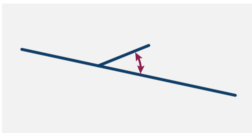
Performing
- Identify angles (static and dynamic) as a measurement between two lines.
- Identify angles (static and dynamic) as a measurement between two planes.

Classifying
- Classify angles as bigger or smaller than one, two or three right angles.
- Order a selection of angles by size.

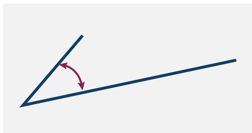
Representing
- Represent physical angles using, for example: in 2D two strips of card and a split pin or a bendy straw; in 3D a folded paper wedge; so that these can then be reproduced on paper.
- Use an arc to label an angle in a diagram.

Analysing
- Compare the size of given angles, for example: by eye; by physically overlapping them; or by using other equipment.
- Consider the (non)-implication of the size of the arms of an angle or arc marking an angle.
- Recognise the different situations that form angles, for example:

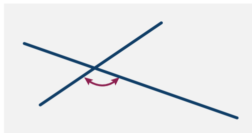
  doors (I hinge);

  books (V hinge);

  scissors (X hinge);

  corners (of shapes, of rooms); wedges; ridges; arrowheads; pencil points; torch beams; oriental fans; a garden water spray.

Modelling
- Model, explore and explain ideas about angles using physical models: for example, using two strips of card and a split pin; a folded paper wedge; or a bendy straw.
- Model an angle between two planes as the angle between two lines in 2D.

Leads out to

lea0.1	by theme Comparing
ang3.2	by theme Angles
ang4.1	by theme Angles
ang1.1	by theme Angles
mea0.5	by theme Angles

Glossary

This node references 2D, angle, arc, bend, corner, diagram, difference, line, measure, model, plane, point, right angle and shape.

Research

This node is linked to

- Blanton, M. L., & Knuth, E. (2009). *Project LEAP: Learning Through Early Algebra Progression*. Wisconsin Center for Education Research at the School of Education, University of Wisconsin-Madison.
- Browning, C. A., Garza-Kling, G., & Hill Sundling, E. (2007). What's Your Angle on Angles? *Teaching Children Mathematics, 14*(5), 283–287.
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- Wilson, P., & Adams, V. (1992). A Dynamic Way to Teach Angle and Angle Measure. *The Arithmetic Teacher, 39*(5), 6–13.

mea0.5 Comparing measures

Type

Landmark waypoint.

What

Recognise how objects and shapes can be compared in a mathematical sense, specifically in terms of angle, length, area, volume and weight (referred to as mass once units have been introduced).

Identify measurable attributes of shapes and objects; 2D shapes have a perimeter (a length), angles and enclosed area, 3D shapes have edges (length), faces (areas), angles (on the faces) and an enclosed volume. They also have a weight.

Recognise the similarities and differences in the actions carried out in order to directly compare in each measure.

Recognise how some actions, including decomposition, do not change the overall angle, length, area, volume or weight.

Why

To develop the understanding of length, area, volume and weight as comparable attributes in objects and shapes;

to develop the mental ability to compare objects;

To begin to understand which transformations can be performed such that the comparable attribute remains unchanged.

Student Actions

Performing
- Directly compare the angle, length, area or volume of objects and shapes (where appropriate).
- Recognise and identify lengths, areas, volumes and angles that can be directly compared.
- Use comparative language, such as: taller, shorter, wider, narrower, longer, shorter, larger, smaller.

Classifying
- Identify objects/shapes with a larger or smaller measure (angle, length, area or volume).

Representing
- Understand, use and justify the symbols: >, <, and = between objects/shapes to represent comparisons.

Analysing
- Compare and make connections between the different methods of direct comparison of angle, length, area and volume.

Solving
- Use the logic of if a<b and b<c then a<c in comparing angle, length, area and volume.

Leads out to

mwd0.2	by theme Mass, weight and density
lea0.6	by theme Relationships
ang1.1	by theme Comparing
are1.1	by theme Comparing
vol0.9	by theme Comparing
vol1.3	by theme Comparing
ppl1.1	by theme Comparing
mea0.9	Measuring

Glossary

This node references 2D, 3D, angle, area, attribute, decomposition, difference, edge, face, length, mass, measure, perimeter, shape, symbol, unit, volume and weight.

Research

This node is linked to

- Blanton, M. L., & Knuth, E. (2009). *Project LEAP: Learning Through Early Algebra Progression*. Wisconsin Center for Education Research at the School of Education, University of Wisconsin-Madison.
- van den Heuvel-Panhuizen, M., & Buys, K. (Eds.). (2004). *Young Children Learn Measurement and Geometry*. Freudenthal Institute.

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