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Ontology

Structure and meaning in the Cambridge Mathematics Framework

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Ontology: Structure and meaning in the Cambridge Mathematics Framework

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Introduction

For the Cambridge Mathematics (CM) Framework to support coherence in mathematics learning it needs its own internal consistency of meaning, which must also be consistent with the nature of the mathematical ideas being represented. That consistency has been established and refined as part of our design process[1] and is recorded in our *ontology*, a set of agreed guidelines for how mathematical ideas can be expressed and related to one another in the CM Framework. The ontology also allows us to determine structures for add-on modules in which we can store and work with content outside of the CM Framework, mapping it onto elements of the CM Framework as needed. With guidelines in place, it becomes possible for Framework authors and users to share an understanding of the ideas, connections and structures represented in the CM Framework, and to explore their implications for curriculum design, resource design and professional development. In this paper, we present a description of the CM Framework ontology and how it has been developed. *Research Summaries* from the CM Framework are provided which demonstrate how content can be expressed and visualized according to the ontology.

The CM Framework ontology described here is directly implemented in the structure of the database itself and in the design of the authoring tools underlying the CM Framework and other add-on modules. It is designed to support rather than automate human decision-making and we have created it through qualitative processes of observation, description and discussion so that it can be effective for this purpose. This distinguishes it from quantitatively-derived ontologies more typical of a knowledge management systems approach in computer science.

Structure of the CM Framework

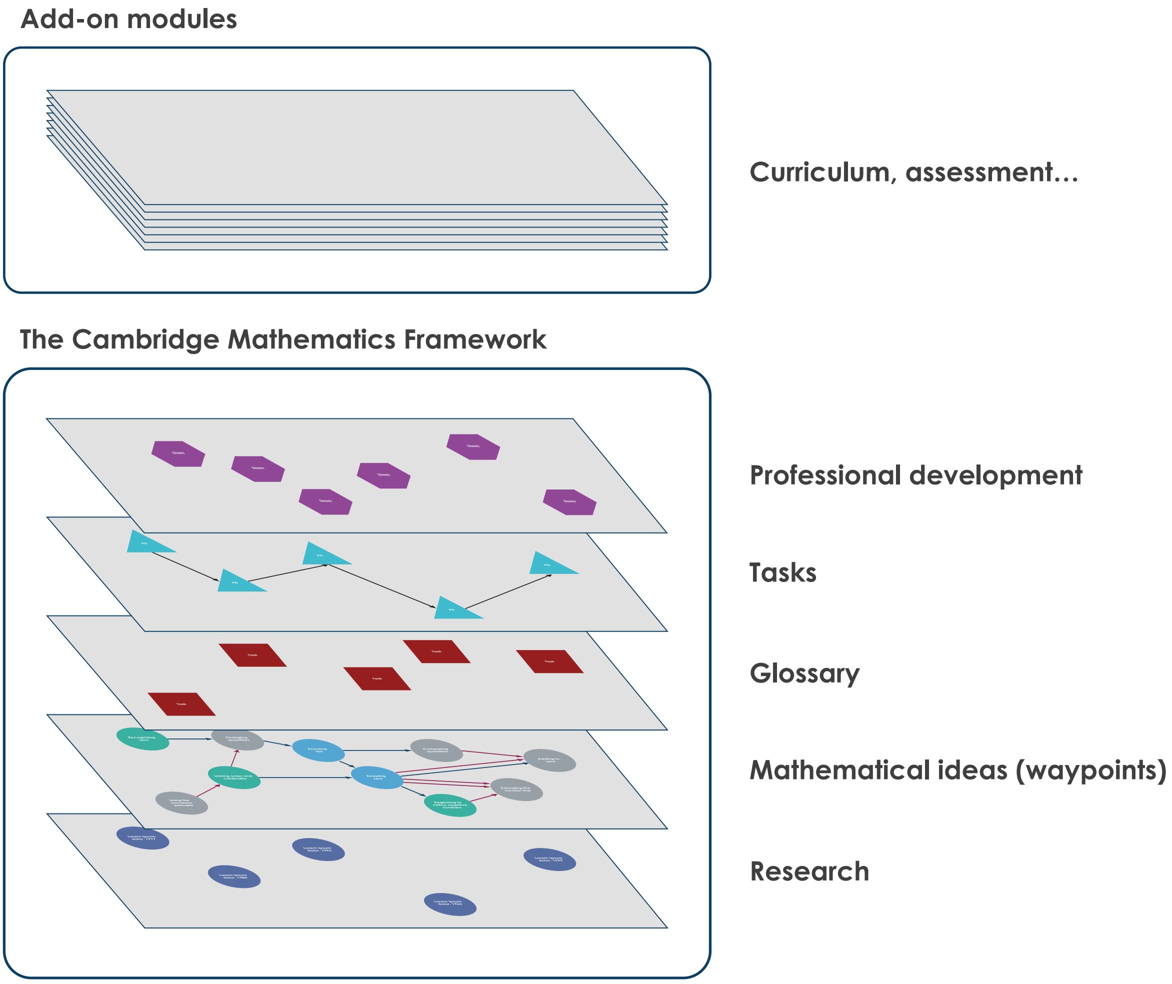
The Cambridge Mathematics Framework treats mathematics as a connected web of ideas in which different meanings can be found at different levels of organisation. This web is built in a graph database[2], in which the mathematical ideas are expressed as nodes (points) and relationships between ideas are expressed as edges (connections between points). We have developed a web-based platform called CMF Nexus (Stevens et al., 2019) with tools which allow us to search, filter and visualise the ideas expressed in the CM Framework and view different levels and types of information as connected layers[1]. Currently we are using these tools to design, author and evaluate the CM Framework and in the future they will also form the basis for a set of tools that others will use to work with it.

[1] Described in Methodology: Research-informed design (Jameson, 2019b)
[2] Neo4j

The structure of the database and the tools for working with the CM Framework have grown and changed as we have developed the structure of the CM Framework itself. This process of refinement continues, and is influenced by continuing literature review, feedback from collaborators and pilot test cases and needs that become apparent through working with the design tools. Refining our ways of working with the CM Framework also feeds back into refining the ontology itself.

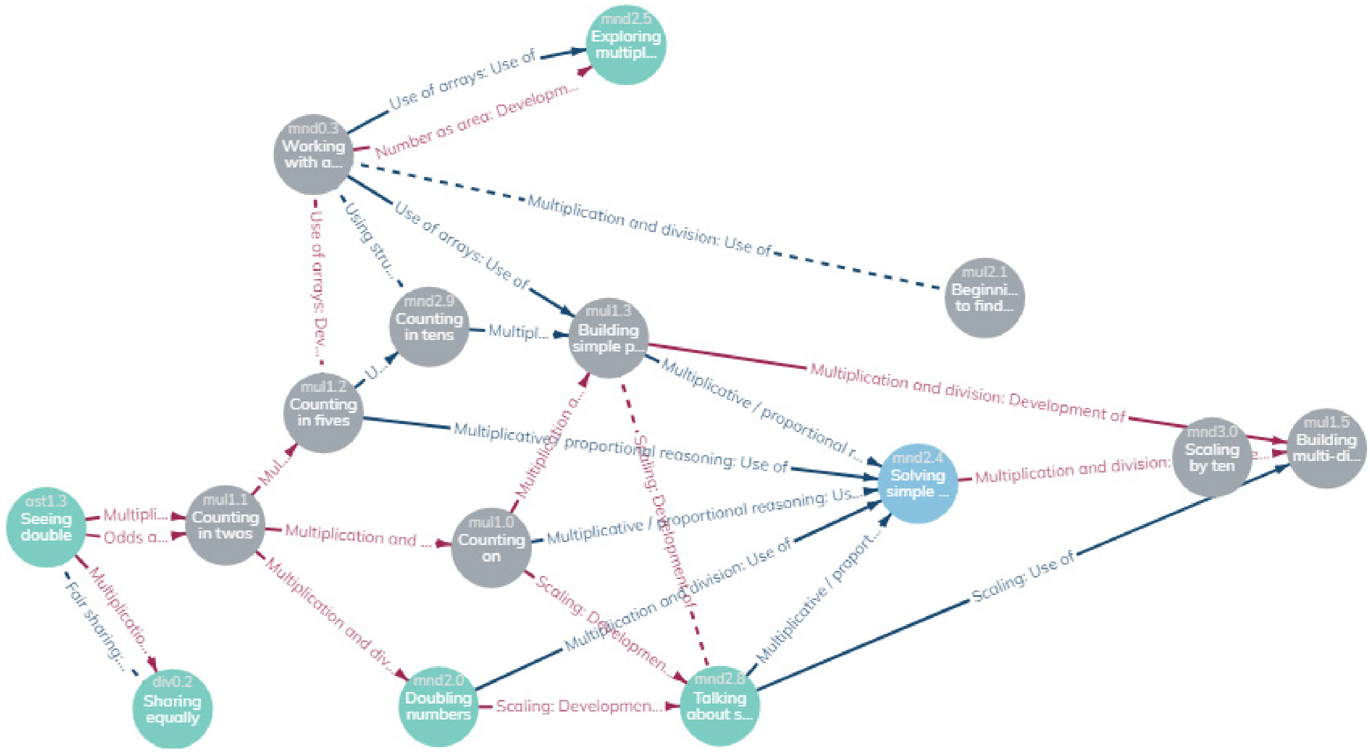
Below we describe the *what*, *how*, and *why* of the ideas and relationships expressed in the Cambridge Mathematics Framework. Figure 1 illustrates *layers* as a way of imagining how different components of the CM Framework can be linked together and how other add-on modules outside of the CM Framework can be mapped to specific Framework components.

Figure 1: Different layers within the Cambridge Mathematics Framework and external add-on modules



Waypoints (Mathematical Ideas layer)

Figure 2: Example of a set of waypoints in the mathematical ideas layer



The Mathematical Ideas layer is the CM Framework layer where we describe mathematical ideas and relationships. The nodes in this layer are waypoints, defined as ‘places where learners acquire knowledge, familiarity or expertise’. This definition is based on characterisations of learning sequences by Michener (Michener, 1978) and Swan (Swan, 2014, 2015). Each waypoint contains a summary of the mathematical idea (the ‘what’) and its rationale for inclusion and its part in the wider narrative (the ‘why’), and lists examples of ‘student actions’ that would give students the opportunity to experience the mathematics in meaningful ways, as described in more detail below. All waypoints in the CM Framework have the above characteristics, but there are also two special roles waypoints can play:

- *Exploratory waypoints* (shown in green in Fig. 2) often come at the beginning of a ‘theme’ (a connection between waypoints, defined below) and indicate a place where ideas can be played with in a less formal or more playful way, as part of building mathematical intuition.
- *At landmark waypoints* (shown in blue), ideas are brought together such that the whole experience may seem greater than the sum of its parts.

Single waypoints may involve different numbers of mathematical concepts, processes, etc., which remain united because the way a student engages with an idea may be more conceptual or procedural depending on that particular student's individual trajectory. These paths indicate connected dependencies and constellations of ideas for the consideration of those who are planning a teaching order, deciding how much time to spend on different topics or deciding when students have sufficiently developed an idea, because these decisions must be made closer to the curriculum context to which they apply. This structure is intended to provide as much support as possible while also remaining flexible enough for that support to be relevant in different contexts. Thus, while the map is fairly conceptual, it is not a direct map of concepts. Likewise, the paths formed by connected waypoints do not represent student trajectories, 'best paths' or teaching orders, and we do not propose any timings within the map.

Visualising and ordering waypoints

Figure 2 provides an example of how we visualise a subset of waypoints as a map. The waypoints shown here are connected to a larger network of waypoints, but the themes shown are only those which connect the visible group of waypoints. Because we are representing the ways in which mathematical ideas contribute to one another, we have chosen to give the map a left-right ordering. In Figure 2, the waypoints towards the left contribute to those they lead into on the right; or, in other words, the waypoints towards the right 'depend on' the waypoints that come before them on the left. In our ontology such dependencies are not absolute; there are more waypoints in the CM Framework than a single curriculum could cover, and there are often multiple alternative routes to a particular mathematical idea.

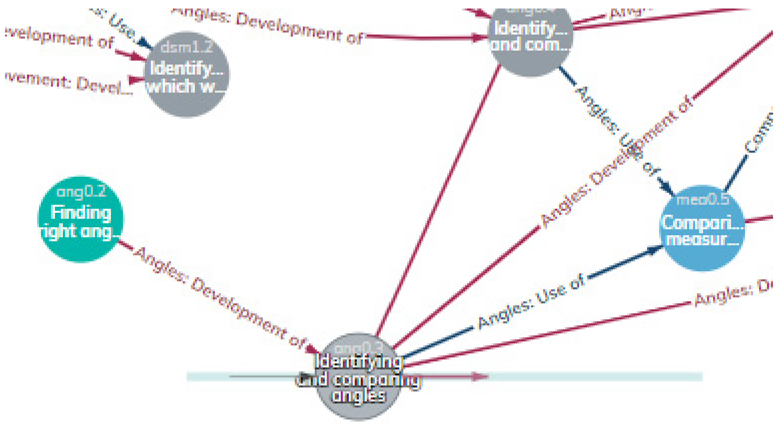
Moving from left to right in the map is not intended to represent the trajectory students would take. Rather, the map can help designers and teachers to keep the conceptual picture in mind so that they can determine their preferred routes and timings. For example, it may be pedagogically useful to go back and forth, jump around, revisit and sometimes skip some waypoints to better emphasise others.

We have developed ordering features of our authoring tools which help us to manage the complexity of our maps and highlight choices to be considered:

- *Starting point for left-right ordering*: Although we can adjust ordering manually, by default CMF Nexus will initially display waypoints in a tentative order from left to right, based on all the ordered relationships between them in the map. We can choose whether this initial ordering will be relative to a subset of waypoints or the entire network. This is only a starting point, as the writing team can then adjust the layout as they develop a set of waypoints.

- *Placement bar*: Once waypoints have been displayed in this initial order, writers can then manually position them within approximate ordering zones based on the lengths of the paths 'leading in' to them in that subset of the network. In the example shown in Figure 3, the waypoint 'Identifying and comparing angles' can be placed equivalently (at least with respect to dependencies) anywhere along the left-to-right extent of the region indicated by the thick light-blue line. The writer creating that map could therefore change its position to anywhere within that horizontal zone.

Figure 3: Example of the placement bar feature: a light blue bar suggests a left-right ordering zone for the waypoint 'Identifying and comparing angles'



Waypoints and design principles: connected understanding and motivating experiences

Our reasons for defining waypoints in this way are tied to our design principles, particularly *connected understanding* and *motivating experiences*. Below, we describe how our perspectives on the role of connected understanding and motivating experiences in building understanding of mathematics have influenced the design of waypoints.

Mathematicians, learning designers and mathematics education researchers each have perspectives on the development of understanding in mathematics that we have found helpful. A broad functional distinction between 'understanding' or 'not understanding' some mathematics is held roughly in common. If someone does not understand some mathematics, even if they 'know' it as a fact or a process, they will not be able to use it flexibly or powerfully (Usiskin, 2015). The NCTM Principles and Standards for School Mathematics framework distinguishes between conceptual understanding, factual knowledge and procedural fluency as three interdependent components of *proficiency*: the ability to use mathematics well (Mathematics, 2000). Hiebert & Carpenter (1992) note that understanding allows students to be inventive and intuitive with mathematics, to remember it more effectively and efficiently, to transfer what they know to new contexts and to build positive and productive beliefs about the domain.

If this is what understanding achieves, how does it develop, and how do students use it to build their own sense of the structure of mathematics? Thurston (1990) describes the structure of mathematics as tall, broad and connected. It is 'tall' since concepts tend to build on other concepts; the structure of this is "like a scaffolding, with many interconnected supports" (p. 2). It is 'broad' because it involves many interconnected concepts and this breadth is necessary to support the height of the tall structure – without a broad enough base, individuals' understanding would have no support for growing 'taller' past a certain point. This connected structure makes it 'compressible,' allowing students and mathematicians to access and use effectively the web of conceptual understanding and practices they have built in size and complexity over time (Tall, 2013; Thurston, 1990).

All of this suggests that the development and growth of understanding is made possible by a web-like scaffolding structure that each student builds for themselves, with classroom experiences playing a large part. In deciding what this means for the design of the CM Framework, we draw on existing research for ideas of both the connections made within the learning mind and connections within the domain of mathematics as a logical system, while remaining aware of the limitations of what may be currently 'known' about either for any given area of mathematics. There is a growing body of empirical work on learning progressions, trajectories or pathways, concerning the connections that students make in building conceptual understanding (Black, Wilson, & Yao, 2011; Daro, Mosher, & Corcoran, 2011; Maloney, Confrey, & Nguyen, 2014). Michener (1978) developed a framework for mapping mathematics understanding that expresses concepts in directed graphs like ours; while it was developed rigorously it has so far been actively applied to limited areas of the mathematics learning domain. These studies provide invaluable insight but so far tend to be concentrated in certain areas of the mathematics education literature (concepts commonly introduced in pre-primary and primary years), and the process of producing and testing conjectures is intensive and slow-moving relative to the time frame for our project. While this research, where it exists, influences our work, and while a great deal of what there is can be applied to some topic areas, there is still not sufficient coverage to form the basis of our design.

When we take a step back from the idea of directly mapping the paths that students take, we can look at other ways that connections are described within the domain. Hiebert & Carpenter (1992) describe students forming "rich networks of knowledge" (p. 74) to which elements are added *from the time they* *are first experienced* rather than after students are proficient. Students can then benefit from considering those ideas together with related concepts or processes that offer experience of the concepts in action (Denvir & Brown, 1986; Skemp, 1979).

Although we cannot map these rich networks literally, we can still focus on identifying elements of the domain and ways that relationships between those elements are characterised in existing research. In this way we can avoid making claims, in the absence of data, about which pathways students are more likely to take, but we can provide support to designers and teachers who may have their own working hypotheses or guidelines. By linking existing research on learning trajectories to such a structure, we can also show where more research on learning trajectories might make a difference to decision-making in design and teaching.

Waypoints and student actions

To enhance our descriptions of mathematical ideas at waypoints we give examples of *student actions*, the kinds of things that students might do to help them build an understanding of the 'what' and the 'why'. Student actions are derived from a framework for the design of tasks which support building conceptual understanding and procedural fluency in mathematics (Swan, 2014, 2015). Swan's purpose in creating his original framework was to expand the ways in which mathematics curricula could be interpreted in the design of curriculum resources and assessments (Swan, 2014), and even ways in which curricula could be designed (Swan, n.d.).

This focus on student experiences also provides common ground for shared understanding and communication between designer, teacher and researcher roles, both facilitating individual work and helping to coordinate it with the work of others. In many jurisdictions, communication about connections that are important in different contexts is infrequent in education settings. This is partly because there is rarely much overlap between professional communities involved in various stages of mathematics education and practice (e.g. high school, undergraduate, mathematical research, applied mathematics, computer science, statistics, etc.) (Thurston, 1990) and there are important differences between the ways these communities consider mathematics (Usiskin, 2015).

Swan's original task design framework is a task taxonomy (taskonomy?) in which these actions are associated with different categories of goals for mathematical practices and ways of thinking: mathematical literacy, solving problems, reasoning and communicating, conceptual understanding, procedural fluency and factual recall. Swan tied these to 'task genres' or types of activities which might give students experience with particular mathematical practices, and to examples of these activities.

What we frame as student actions Swan presented as student products – sample activities paired with a description of what students would produce as a result of successfully doing these activities (Swan, 2014, 2015). Our application of Swan's framework to curriculum design has precedent in his own work, in which he titled a variation of it "Framework for designing a balanced mathematics curriculum" (Swan, n.d., p.1).

In the Cambridge Mathematics Framework, we intentionally focus on examples of actions that students do rather than things they produce. Our aim is for the sum total of structure, descriptions and student actions in a set of waypoints to convey information to CM Framework users about *how* students might act as part of the process of meeting learning goals. We believe that this will help the CM Framework to be a more flexible tool, because we recognise that learning goals must be specific to a curriculum, resource, or scheme of work and will be specified by users of the CM Framework rather than by us.

This focus on student actions can also help us to avoid implying more than we should, based on the evidence we have, about the order in which students can engage with specific ideas in a curriculum, as students may engage with different aspects of ideas in different ways at different times. Rigorous research leading to the development of student learning trajectories has been carried out and applied to decisions about ordering learning objectives in mathematics curricula (Confrey, Maloney, & Corley, 2014; Daro et al., 2011; Maloney et al., 2014). The CM Framework, in contrast, is designed to support curriculum designers in noticing whether their work is consistent with the broader logic of conceptual ordering. Thus, in order for us to make use of the range of evidence that is available across the breadth of the mathematics learning domain that the CM Framework aims to cover, we make waypoints with student actions the basis for the structure.

Table 1: Student actions in the Cambridge Mathematics Framework, adapted from Swan (2014, 2015)

Category	Student actions	Description
Procedural Fluency	Performing	Memorise and rehearse
Conceptual Understanding	Classifying	Sort, classify, define and deduce
	Representing	Describe, interpret and translate
	Analysing	Explore structure, variation, connections
	Arguing	Test, justify and prove conjectures
	Estimating	Use a sense of magnitude to make sensible predictions
Problem Solving	Modelling	Formulate models and problems
	Solving	Employ strategies to solve a problem
	Critiquing	Interpret and evaluate solutions and strategies

Swan (2014) characterises the theoretical influences involved in his selection of these actions as mainly social constructivist, with underlying ideas of understanding in mathematics and integration of activity theory as laid out in Sierpinska (1994). He names specific areas of the literature that he has drawn on in prioritising the components of the original task design framework, based on his goal for expanding curriculum interpretation in task design:

> Concepts and strategies are co-created as language and symbols are appropriated and internalized (Bakhtin 1981; Vygotsky 1996). The following principles are perhaps the most important:
>
> - Use formative assessment; build on and adapt lessons to the knowledge that students already have (Black & Harrison 2002; Black, et al. 2003; Black & Wiliam 1998; Black, et al. 1999);
> - Develop mathematical language through communicative activities that encourage dialogic talk (Ahmed 1987; Alexander 2006, 2008; Mercer 1995);
> - Focus directly on either specific conceptual obstacles or processes (Bell, 1993; Wigley, 1994) and create surprise, tension and cognitive conflict that may be resolved through discussion (Brousseau 1997);
> - Create connections between topics both within and beyond mathematics and with the real world. Use multiple representations (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997);
> - Foster peer assessment by using tasks that allow students to shift roles and explain and support one another (Bell, Swan, Crust, & Shannon, 1993b);
> - Use tasks and questions that promote explanation, application and synthesis rather than mere recall (Bills et al. 2004; Watson & Mason 1998).
>
> (Swan, 2014, p. 16)

We have incorporated student actions to help designers and teachers bridge the distance between the intended, interpreted and enacted curriculum[3]. By adapting the Swan framework, our intention has been that everybody will have a place to anchor their current understanding; to understand what is in the CM Framework according to what they recognise. Someone with more conceptual but less didactic experience might first relate to a map or the waypoint ‘what’ and ‘why.’ Someone with more experience in teaching and/or task design might find that student actions are a useful way to engage with the structure of the map or the rest of the content of the waypoints.

[3] In a Research Summary, the student actions provided for a waypoint can be seen with the other properties of that waypoint by clicking on it in the dynamic map. In the example Research Summary provided, all of this waypoint content can be found at the end.

Theme edges (Mathematical Ideas layer)

Waypoints are connected by edges which represent some relationship between them. We call these relationships *themes*, each of which is named according to the concept/skill/procedure we believe the relationship to represent (e.g. 3D Shapes, Inference, etc.) and whether the connection between the waypoints is best described as the *development* of a concept/skill/procedure or as the *use* of a concept/skill/procedure. 'Development of' edges are *directed,* meaning that the edge has a single arrow pointing from one node to another node which is furthering along a sequence of development of that idea. 'Use of' edges are *undirected* meaning the edge does not imply a sequence of development. We treat concepts, skills, and procedures together as aspects of mathematical ideas as they might be experienced by different students at different times. These two types of themes can be seen in the attached Research Summary. For example, in the *Early angle conception and measurement* Research Summary, an exploratory waypoint leads to the waypoint "Identifying and comparing angles" by a theme called "Angles" (Development of), and in turn the "Identifying and comparing angles" waypoint is connected to "Comparing measures" by the "Angles" theme (Use of). Other layers can connect to the waypoints layer as well; for example, in the research layer, research edges link research nodes (see below) to specific waypoints or groups of waypoints. However, in the rest of this section we will focus on themes within the Mathematical Ideas Layer.

There are two categories of decisions which emerge as the writing team considers whether to express an idea in the CM Framework as a theme edge:

1. **Local topic-area scale: Should an idea be expressed as a landmark waypoint or as a theme edge that connects a series of waypoints, or both?** Many themes are mostly local to a particular topic area and may culminate in a related landmark waypoint. The difference between the theme edge and the landmark waypoint is partly due to the structural role of edges (theme edges keep track of *why* a series of waypoints helps build up to the mathematical experience described at a landmark waypoint) and partly because landmark waypoints may be *more than* a single mathematical idea, with multiple themes contributing.

2. **Global scale: Should an idea be expressed as a theme edge between waypoints or is it too broadly distributed for these edges to be meaningful?** We have created some themes that are more global, linking waypoints in multiple areas of the CM Framework, and some ideas are even more widespread across many areas of the CM Framework. Functionally, some of these might still work well as themes. Others may not be most appropriately expressed as themes because they do not play a useful structural linking role, since either they would have to be almost everywhere or they do not otherwise connect waypoints to form a local, topic-specific set of paths within the CM Framework. For these ideas, we use tags to keep track of waypoints they pertain to and will decide in the future how best to express these ideas in a useful way.

Once a theme is first used by a member of the writing team, it becomes part of a list that can be used by others, with discussion of the meaning of the theme and how to use it appropriately. Particular themes may be edited as work on related waypoints progresses and opportunities emerge to evaluate their role.

The CM Framework is a construct built by individual authors, so their decisions about themes and waypoints determine which mathematical ideas are expressed in the CM Framework and how. It would be possible for others to make different choices given different – or even very similar – considerations. This is why we use Research Summaries to explain specific decisions about the creation and structuring of individual themes.

Research nodes and edges (Research layer)

Our design is informed by

- research evidence from literature reviews, and
- collaboration with researchers and curriculum designers with experience in particular areas of the literature.

We make as much use of this type of information as possible and the vast majority of waypoints and themes are linked to these sources. However, in some topic areas, less research is available and/or research implications may be more open to interpretation. In such cases, some decisions we make are unsubstantiated other than by knowledgeable practitioners, including the team's own experience.

Our Research layer contains:

- *Research Summaries*, which record the basis for the contents and structure for a specific selection of nodes and edges in the Mathematical Ideas layer,
- *Research nodes*, which correspond to specific sources and contain metadata characterising those sources, and
- *Research edges*, which connect research nodes to research summaries or directly to waypoints as appropriate. They also describe whether a source has been an important or a secondary influence on the waypoint or Research Summary it links to.

As sources are used in writing the CM Framework they are entered into the CM Framework database as research nodes and linked to related waypoint(s). Each source's contribution is described in one or more Research Summaries. Research nodes and Research Summaries, together with corresponding sets of nodes, make up the research base that provides the foundation for collaborative discussions and expert review of specific areas of the CM Framework. The Research layer contributes to our *transparency* design principle by allowing us to make the reasons for our decisions explicit so that they can be critiqued, adapted and used according to the judgement and context of the designer or teacher who ultimately considers them.

Building the research base

Areas of the CM Framework are written on the basis of literature review and consultation with researchers and subject experts whose work is generally accepted by at least some portion of the mathematics education community. The research we review has a fundamental influence on the structure and content of the CM Framework and our research methods determine what research we review. Due to their importance in our work, we have described our research methods in detail in Methodology: Building the research base (Jameson, 2019a).

Glossary nodes and edges (Glossary layer)

Key mathematical terms or phrases are defined in glossary nodes which are linked to the waypoints in which they appear. These allow us to access definitions while looking at the CM Framework and surface the parts of the CM Framework which are linked to a particular term[4].

Other CM Framework layers in development

Other planned CM Framework components include task design and professional development layers. We are collaborating on small projects which serve as case studies to inform the design and development of these CM Framework layers.

In the task layer, each task node will describe either the detail of a classroom activity (including formative assessment) or a summative assessment activity, or both. Some of the tasks will be linked to just one waypoint while others may span two or more, including across mathematical topics. This will exemplify the interconnectedness of mathematics in a different way. Initial research and planning for the professional development layer is ongoing.

[4] The design and evaluation of the glossary are described further in Methodology: Research-informed design (Jameson, 2019b), and Methodology: Glossary app (Majewska, 2019) respectively.

Add-on modules outside of the CM Framework

We are discussing and building examples of other layers consisting of other kinds of nodes which are linked back to the underlying waypoints and research base, each for a particular purpose. Currently these purposes include support for mapping to inform curriculum development and assessment. We are also experimenting with different categorical groupings of waypoints for various purposes – for example, displaying waypoints which make up large subdomains, or displaying all landmark nodes as a map in their own layer, condensing the waypoints and themes between them into simplified paths. Such grouping layers are not constituent parts of the CM Framework in their own right but help us to analyse and interpret it. We describe the curriculum and landmarks layers below; other layers will be created as we trial additional uses of the CM Framework.

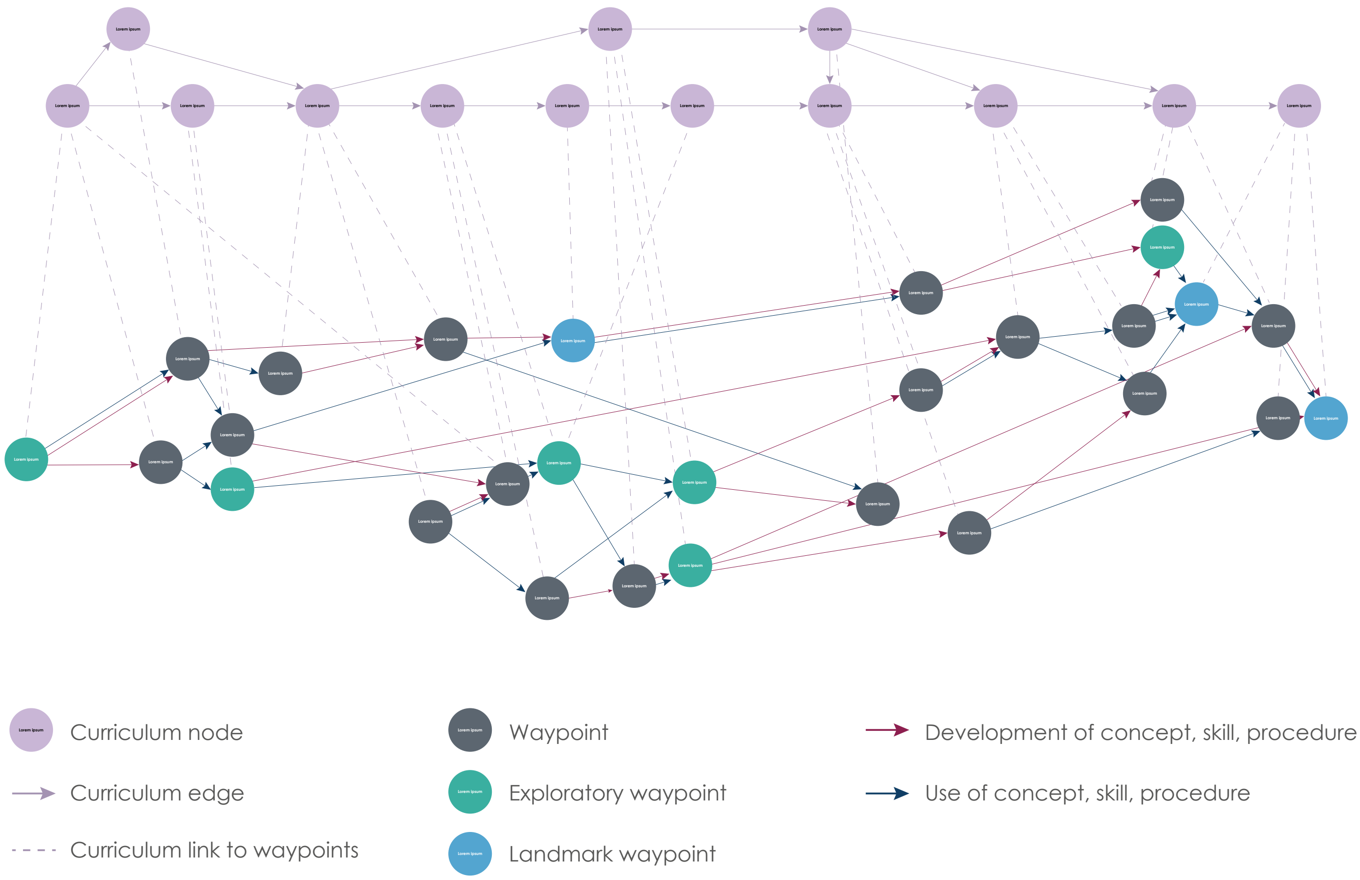
Curriculum layer

We have created a curriculum layer as part of ongoing case studies in which we and our collaborators are exploring the use of the CM Framework to design curricula. Currently the curriculum layer comprises two feature types.

- *Curriculum statement nodes* represent high-level curriculum statements for a particular curriculum under development, which would more traditionally be presented as a list or a table. We link these to waypoints which are reasonably related to them in our Mathematical Ideas layer. The end result of this linking process is a set of curriculum statement nodes with a network of selected waypoints underneath. These waypoints remain connected to the larger network of waypoints but can also be treated like a subnetwork, with the ordering of waypoints determined by the ordering of the directed edges, either in the subnetwork or the whole network.
- *Curriculum statement edges* are created automatically based on the connectivity of linked waypoints in the Mathematical Ideas layer. Curriculum statement nodes can then be tentatively ordered on the basis of this connectivity.

The curriculum layer helps to bridge the detail of the Mathematical Ideas layer and the curriculum design process, in which decisions must be made from among those options. The final product is a set of curriculum statements ordered according to the context and goals for a particular curriculum, which can form the backbone of more detailed timing and schemes of work/scope-and-sequence planning.

Figure 4: An example of the curriculum layer, showing curriculum statement nodes and edges (pink, top) and waypoints and theme edges (all non-pink, below)

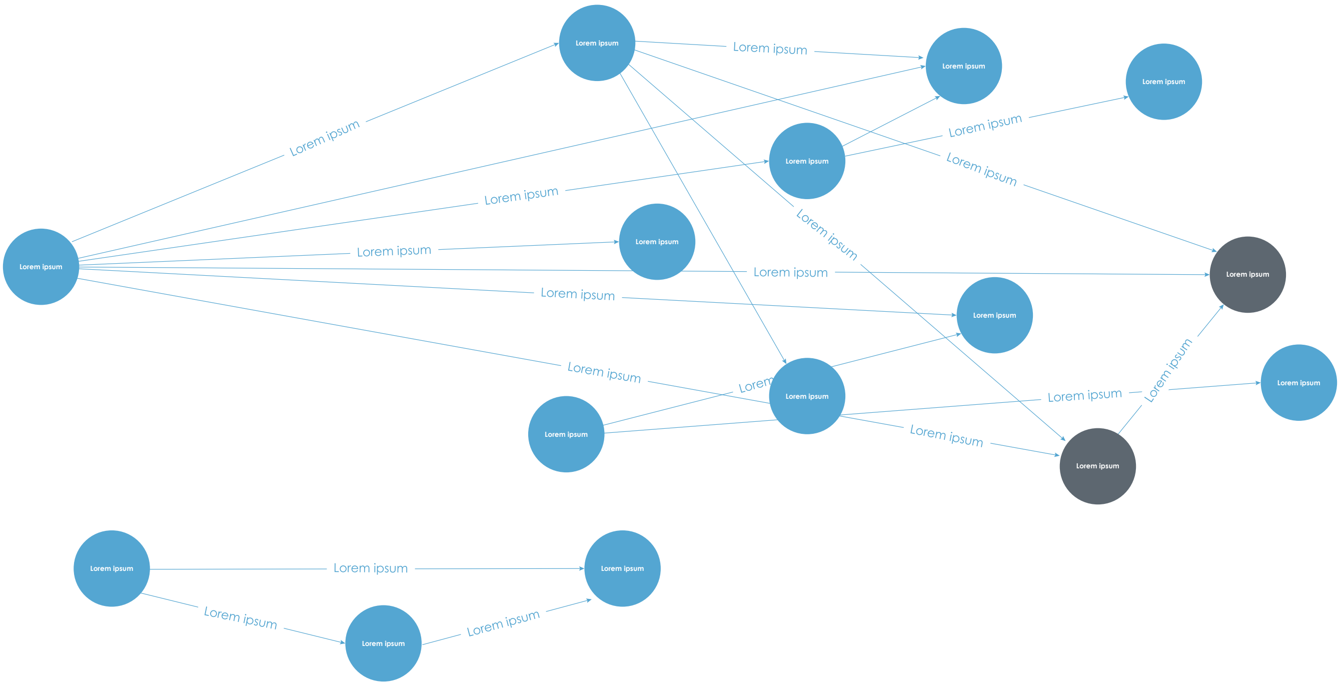


Landmarks layer

In order to get a sense of the bigger picture, we can zoom out from the detail of the waypoints to focus on landmark waypoints as a map in the *Landmarks layer*. *Landmark edges* represent the shortest path of waypoints connecting two landmarks and each path lists the collection of themes it comprises.

Figure 5 on next page

Figure 5: A selection of landmarks in the Landmarks layer



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Summary of design principles[5] and design features supported by the ontology

Design principles	Design features
Coherence	• Curricular coherence: Connecting mathematical ideas to other layers corresponding to different aspects of interpretation and enactment (tasks, teacher professional development, etc.) • Domain coherence: Using the Swan Task Design Framework to characterise waypoints, using waypoint types to signal special roles of certain waypoints; also see *connected understanding*
Connected understanding	• Network structure • Particular specifications for relationships in the ontology
Motivating experiences	• Using waypoints instead of learning objectives leaves room to express important experiences of mathematics without constraining the path through them according to considerations we cannot anticipate or data we do not have
Transparency	• Research nodes • Research Summaries
Flexibility	• Multiple relationships may offer choices in designing paths • Waypoints as experiences keep expertise of users in the design loop • Recognising that different jurisdictions have different criteria and constraints for their curricula • Recognising that a curriculum is rarely designed from scratch or implemented as a major discontinuity, so the CM Framework must support coherence at small scales as well as larger ones • Live digital graph database can easily incorporate additions and updates to content and structure

[5] These design principles are described in A Manifesto for Cambridge Mathematics (McClure, 2015) and An update.... (Cambridge Mathematics, 2018) Development and refinement of these design principles is described in Methodology: Research-informed design (Jameson, 2019b).

Developing and using the ontology

Developing the ontology

We draw on design methods in education that have been developed and refined within design research methodology for over twenty years (Cobb, Confrey, Lehrer, Schauble, & others, 2003; McKenney & Reeves, 2012). Particular aspects of design research that make some of its methods appropriate for us (apart from a general focus on design) include: linking specific design priorities and choices to theory; using initial design work to develop design principles that inform ongoing work; going through iterative cycles of design in which feedback on work in progress is incorporated into new design versions and practices; and participation in design by experts in multiple relevant communities (Barab & Squire, 2004; McKenney & Reeves, 2012; van den Akker, Gravemeijer, McKenney, & Nieveen, 2006). We have integrated ontology design into our overall design process[6], and in many respects ontology design is a microcosm of the larger design process.

Our ontology design process has followed the general cycles of ontology development described in Fürst, Leclère, & Trichet (2003, p. 80). We started with an informal conception of the mathematics learning and relationships we wanted to model in a few foundational areas of the CM Framework, like *Counting*. We built an initial conceptual model and revised it many times based on discussions between writers working in different areas. We started to agree on more structural features and ways of characterising mathematical ideas that could remain consistently meaningful when applied to different areas; this became a rough ontology, which was then subject to further revision as new issues emerged that needed to be resolved.

In later stages of this process, we formalised the ontology in the CM Framework database, moving from paper and spreadsheets to a consistently applied custom format. This format is still being actively revised and updated as issues arise and consensus begins to form around existing questions. Features of the CM Framework are now stored in CMF Nexus, which is designed around our ontology[5]. Components of the CM Framework which name and describe mathematical ideas, the *entities* (primary units) of the ontology, are *nodes*, and relationships between them are *edges* in the graph.

[6] Described further in Methodology: Research-informed design (Jameson, 2019b).

Using CM Framework authoring tools to refine the ontology

An ideal ontology would define entities and relationships in such a way that there was no ambiguity or overlap, with a minimum of real-world messiness unaccounted for. When there is not enough information available to do this, an ontology may remain in an intermediate state of development, with a more formal part consisting of clear or at least consensual structures and definitions, and a more informal part consisting of elements "in development" whose structure or meaning may not yet be agreed. Once a version of a formal representation exists, designers can build interfaces, groupings and hierarchies that make it meaningful and accessible as a shared representation, which then fuels further cycles of development (Fürst et al., 2003).

Our ontology is the result of our subjective interpretations, formed through qualitative processes. This is expressed and manipulated quite differently from a typical knowledge database ontology where the purpose is to automate decision-making or data classification. Our intention is to interpret and express ideas with consistency and meaning so that they can be understood and used, even when they are not part of a curated explanatory document like a Research Summary.

The CM Framework viewing and authoring tools we have developed in CMF Nexus prompt us to adhere to our current ontology. However, we have also built ontology design tools in CMF Nexus so that we can refine the ontology when we want to express something important which doesn't yet "fit" across the structure of the CM Framework. These help us to experiment by trying out new types or categories in the ontology (e.g. new categories of nodes or edges), which can help us show new kinds of ideas, connections or groupings. Some of the features we've experimented with using these tools have been incorporated into the ontology, while others have helped us to agree on the boundaries of what we need to express.

Evaluating the ontology

The CM Framework is a knowledge model structured and defined by our ontology. Models, as simplifications which purposefully emphasise some considerations over others, must be judged relative to their purpose and intended usefulness (Barlas & Carpenter, 1990). In developing the CM Framework ontology, we considered several factors together as suggested by Barlas & Carpenter:

- the purpose of the CM Framework (described on our website[7]),
- the potential usefulness of the CM Framework structure, *for particular uses, with respect to this purpose* (explored in ongoing pilot case studies[7]),
- the perspective of the CM Framework designers on creating and evaluating knowledge models (from our own backgrounds and further research), and
- the relevant perspectives of those evaluating the ontology (described below).

As the ontology started taking shape, we reflected on how it could be used, both by us as designers and by potential users from our perspectives. Once we had produced a usable amount of CM Framework content using the current version of the ontology, we began to gather feedback. We conducted a Delphi study[8] so that people external to the Cambridge Mathematics team, with a depth of experience in national-level mathematics curriculum research and design, could evaluate the trustworthiness of the ontology, applying their depth of mathematics curriculum experience to judging its value. We continue to evaluate the usefulness and trustworthiness of the ontology through ongoing case studies[2].

Similar approaches have been described by framework development projects in other contexts. In a retrospective review of the standards writing process for the *NCTM Principles and Standards of School Mathematics* framework, the writers noted that a set of theoretical perspectives emerged as important influences over time and described their design work as "researchlike" because it also involved collecting, analysing, and incorporating feedback on work in progress (Ferrini-Mundy & Martin, 2003). Of work currently in progress, the UNESCO Institute for Statistics is developing the Reference List & Coding Scheme (RL&CS) Framework, intended to provide rich qualitative support for mapping theory and curricula to assessment frameworks. This project's interpretive approach similarly required the designers to evaluate trustworthiness on the basis of practical value of the construct, as expressed through feedback on work in progress by expert members of the communities that would be making use of it (Cunningham, 2017).

[7] In An update… and Shared perspectives on research… (Jameson, McClure, & Gould, 2018)
[8] Described in Methodology: Formative evaluation (Jameson, 2019c)

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Using the ontology to write the CM Framework

Framework writers review the literature appropriate to each area of the CM Framework[9]. They make choices informed by the ontology about what to express separately or in combination at waypoints and which relationships to other waypoints are important to include. Consistent expression of the ontology when writing waypoints, which is informally checked in writing discussions, has steadily improved.

For a particular Research Summary topic, the team note the apparent consensus or possible lack of agreement between sources with respect to the context of the theoretical stance and questions driving each study. Information from the specific set of choices and influences for each topic is synthesised into a set of nodes and edges and is documented in Research Summaries. The influence of research sources on our work in that topic area is then subject to evaluation and revision through our external review process[9]. Connections between different areas of the CM Framework are made during initial synthesis, discussion with other writers, and sometimes through external review. Over the course of CM Framework development, the team use search and visualisation tools to consider implications of their work and to fuel discussion. Important terms are linked to the glossary where they may be defined based on literature review if they are not already there.

We use a set of web-based tools in CMF Nexus to visualise, create and edit text and structure in the CM Framework database, and to collect and review comments on particular features. The development of these tools has evolved alongside the structure of the CM Framework itself according to the needs of writers and reviewers. We anticipate that our experiences designing and working with these tools will help inform the interfaces eventually used by others to access the CM Framework.

Using the ontology to support a shared frame of reference

The ontology helps us to make ideas explicit and to express them consistently in ways that can be debated, critiqued and refined. This is essential for meaning-making not only among the members of the CM Framework writing team but between the writing team and reviewers, and ultimately between the CM Framework and those who use it in their work. Our initial choices for the ontology were based around the need for shared understanding to improve coherence across curriculum and resource design and enactment. One of the dimensions of the ontology we seek to evaluate is therefore its potential to support the CM Framework as a shared frame of reference.

[9] As described in Methodology: Building the research base (Jameson, 2019a)

Our perspectives on professional knowledge and knowledge-sharing with respect to the CM Framework ontology include models of pedagogical content knowledge from the perspective of designers and teachers (Shulman & Shulman, 2004), models of co-design practices for knowledge representation (Sanders, 2002; Schmidt & Bannon, 1992), individual and social processes of technology-mediated knowledge building (Stahl, 2006), and theories of boundary objects (Star & Griesemer, 1989) and boundary negotiating artifacts (Lee, 2005). Star describes boundary objects as coordinating actions and understanding between groups in spaces where this coordination must take place but existing formal standardisation is not adequate (Boland, 2015); they are therefore potentially less formal or less agreed but nevertheless functional. Boundary objects (Star & Griesemer, 1989, p. 393)…

- "both inhabit several intersecting social worlds…and satisfy the informational requirements of each of them",
- "are both plastic enough to adapt to local needs and the constraints of the several parties employing them, yet robust enough to maintain a common identity across sites",
- "are weakly structured in common use, and become strongly structured in individual-site use",
- "may be abstract or concrete",
- "have different meanings in different social worlds but their structure is common enough to more than one world to make them recognizable, a means of translation", and
- play a key role "in developing and maintaining coherence across intersecting social worlds".

It is important to note that a part of Star's conception of boundary objects is that they "arise over time from durable cooperation among communities of practice" (Lee, 2005, p. 390). Therefore, we do not claim that currently the Cambridge Mathematics Framework is itself a boundary object. However, this theory helps to inform our design goals so that the CM Framework will have the potential to be adopted as such.

Objects which do not arise organically in use by multiple groups may still do some of the work of boundary objects. In particular, our ontology and authoring tools in the CMF Nexus platform currently fit Lee's (2005) description of *prototypes* which "serve as partial or complete representations of the product or process that is being produced". These might act as "boundary objects but also as representations that are necessary to support the understanding of boundary objects" (Lee, 2005, p. 392). For example, we have not yet determined whether the Mathematical Ideas layer or other layers like the Curriculum or

Task layer might be more appropriate to consider as boundary objects depending on how each will be used and who is involved. However, developing and using such layers within the team has helped us to understand what features they might need in order for others to be able to work according to a shared understanding.

Lee also describes *boundary negotiating artifacts*. These are sharable expressions of knowledge used by designers in order to "iteratively coordinate perspectives and to bring disparate communities of practice into alignment, often temporarily, to solve specific design problems that are part of a larger design project" (Lee, 2005, p. 394). In particular, among the types of boundary negotiating artifacts Lee observes, the CM Framework ontology plays a role in

- self-explanation (working out ideas before sharing them),
- inclusion (proposing "new concepts and forms") (Lee, 2005, p. 396),
- compilation (coordinating information and actors), and
- structuring (establishing "ordering principles") (Lee, 2005, p. 398).

The ontology expressed in the CMF Nexus platform is an artifact which allows us, as a design team, to share and negotiate with outside collaborators understanding of what can be and should be represented in the CM Framework and discuss how that might best be achieved. Designing it has helped us to develop our individual expressions of mathematical ideas, to develop a shared structure for our work, to compile that work and make it accessible to others and to recognise when some new feature is needed to express an important principle across the CM Framework.

An example application of the CM Framework ontology

Before the CM Framework itself is released, Research Summaries will provide a useful window on our application of the ontology. Research Summaries in CMF Nexus have an embedded interactive map. A static PDF version is attached: *Early angle conception and measurement*.

Each Research Summary describes the contribution of a set of waypoints to a larger topic by providing:

- a title stating the topic,
- a literature review describing the research informing our interpretation of the topic,
- a map representing the waypoints and connections we have synthesised based on the sources cited and interpretation from consultation and experience,

- a section explaining how our interpretation of the research has led to the waypoints and connections in the map, and
- a bibliography, consisting of a list of references directly cited in the Research Summary and a list of other sources which have informed content in the selection of nodes which are included in the Research Summary.

The example Research Summary is a static snapshot from the CM Framework; in its current state it has been reviewed internally by our team and informally by external collaborators. The contents of the live versions of these Research Summaries within the CM Framework may change in the future due to feedback from the external review process[8] and/or other changes to the relevant set of nodes or additions to the research base.

Summary

We use an ontology to structure the Cambridge Mathematics Framework so that it can be written consistently, understood easily, and used effectively. This structure:

- helps us to align mathematical ideas with curriculum statements, key terms, tasks, assessments and teacher professional development strategies,
- maintains links to explicitly recorded interpretations of the underlying research,
- allows us to share ideas and interpretations among ourselves as a team and with other current and future users of the CM Framework, and
- facilitates digital management, authoring and representation of the CM Framework, which in turn supports the explorations, insights, discussions and explanations which are at the root of the curriculum design process.

A Research Summary, *Early angle conception and measurement*, is provided as an example of the CM Framework content which results from our use of the ontology.

[8] Described in Methodology: Formative evaluation (Jameson, 2019c)

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Appendix A: List of objects and relationships in the ontology

Cambridge Mathematics Framework

Objects (nodes)

- Standard waypoints (properties include ‘what’, ‘why’, ‘student actions’)
- Exploratory waypoints (properties include ‘what’, ‘why’, ‘student actions’)
- Landmark waypoints (properties include ‘what’, ‘why’, ‘student actions’)
- Research nodes (properties include our research source metadata)
- Glossary nodes (properties include terms and definitions)
- Task nodes (to be determined)
- Professional development nodes (to be determined)

Relationships (edges)

- All waypoint-waypoint ‘development of’ (directed or undirected; properties include theme title)
- All waypoint-waypoint ‘use of’ (directed or undirected; properties include theme title)
- Research-waypoint ‘informs’ (properties include strength of influence of that research on the writing of the waypoint)
- Glossary-waypoint ‘is key term in’
- Others to be determined (task, professional development)

Add-on Modules

Objects (nodes)

- Curriculum statements (properties only include title at the moment)
- Others (to be determined)

Relationships (edges)

- Curriculum statement-curriculum statement ‘leads to’
- Waypoint-curriculum statement ‘is part of’
- Landmark node-landmark node ‘leads to by path length’
- Others (to be determined)