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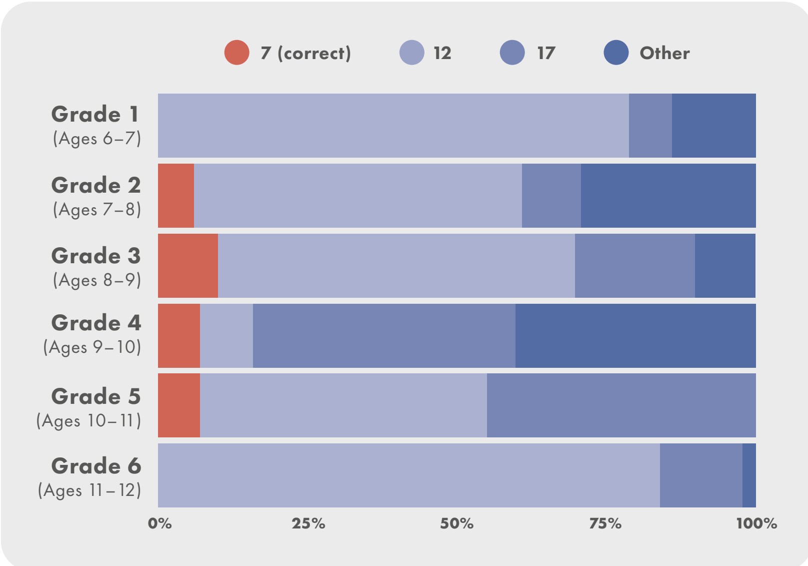
TALKING POINT:

WHAT DOES RESEARCH SUGGEST ABOUT TEACHING AND LEARNING THE EQUAL SIGN?

IN SUMMARY

- Use of the phrase *equal sign* instead of *equals sign* may help to remind teachers and students of its multiple meanings beyond just "perform a calculation"
- Students should have opportunities to encounter multiple properties of equality in their use of the equal sign as part of an intentional focus on its uses
- Developing a solely operational view of the equal sign (reading it as "gives" or "makes") can suggest misconceptions to students; this is currently very common in primary school mathematics in countries such as the UK and the US
- Encouraging students to develop a relational and a substitutive view of the equal sign through explicit instruction helps to develop their mathematical intuition around order, change and equivalence, thus preventing some misconceptions
- Reading the equal sign as "is the same as" helps to promote a relational view of its use
- Developing a sophisticated understanding of the equal sign, including relational and substitutive views, is the basis for successful understanding of algebra

Solutions offered to the question 8 + 4 = □ + 5 by US school pupils at different ages



Adapted from Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, 6(4), 232–236.

1

The equal sign is often used in mathematics classrooms with young students, typically from age 5–7.[1] It is considered straightforward to introduce the *equal sign* (here referred to this way instead of the phase equals sign, to emphasise, as often in the research, multiple meanings beyond "performing a calculation") and once introduced early on for students it is usually not explored more explicitly.[2] However, research suggests "many students at all grade levels have not developed an adequate understanding of the meaning of the equal sign."[3] "Without proper understanding of the equal sign, it is virtually impossible to make sense of how the symbolic language of mathematics works and what it can express."[4] Equality has three fundamental properties: reflexivity ($x = x$, or $8 = 8$), symmetry (if $a = 5$, then $5 = a$) and transitivity (if $a = b$ and $b = c$, then $a = c$) and students should encounter them all as part of an exploration of the equal sign.[5] Students' mathematics performance in general is correlated to their understanding of symbols, and "misuse" of the equal sign can contribute to a variety of mathematical misconceptions, which can be enduring.[1]

IMPLICATIONS: Use of the phrase equal sign instead of equals sign may help to remind teachers and students of its multiple meanings beyond just "perform a calculation"

Students should have opportunities to encounter multiple properties of equality in their use of the equal sign: reflexivity ($x = x$, or $8 = 8$), symmetry (if $a = 5$, then $5 = a$) and transitivity (if $a = b$ and $b = c$, then $a = c$)

More explicit attention to the development of students' understanding of the equal sign from its earliest introduction may help alleviate enduring issues with its use which can create barriers to later learning

2

The most common view of the equal sign is that it is a cue to perform a calculation, often called the *operational* view, where the symbol is often read as "equals," "gives" or "makes." However, this view, if developed in isolation, has been suggested to be a source of later misconceptions that have to be "undone;"[6] for example, struggling to consider equivalence statements such as 4 = 4 or 13 − 0 = 14 − 1. The operational view also may incorrectly suggest that a left to right ordering of a "sum" is part of the meaning of the equal sign, which can lead to issues manipulating arithmetic and later algebraic terms flexibly. "Primary school arithmetic often privileges the operational meaning of the equal sign"[7] and so it is important for teachers to be aware that other meanings of the equal sign, in particular relational ones, should be encountered alongside the operational.[6]

IMPLICATIONS: Developing a solely operational view of the equal sign (reading it as "gives" or "makes") can suggest misconceptions to students; for example, that its use is not symmetrical or it cannot be used for equivalence statements

It is common that primary school mathematics privileges only the operational use of the equal sign

3

Researchers have recognised the need for attention to a *relational* meaning of the equal sign (emphasising mathematical equivalence) in maths curriculum and teaching,[3] developing intuition around order, invariance and parity.[6] Encouraging a relational view of the equal sign from a young age helps to prevent misconceptions, and students who receive explicit instruction on the relational use of the equal sign demonstrate improved performance on equation solving and equal sign understanding.[8] "Minor differences in early input in this way can yield substantial differences in children's understanding of fundamental concepts."[9] A relational view of the equal sign is essential to understanding that transformations performed in the process of solving an equation preserve the equivalence relation[10] and reading the symbol as "is the same as" is a key part of this development of a relational view.[11]

IMPLICATIONS: Encouraging students to develop a relational view of the equal sign helps to develop their mathematical intuition around equivalence, improving their understanding of the equal sign and their performance on equation solving

Reading the equal sign as "is the same as" helps to promote a relational view of its use

4

A sophisticated understanding of the equal sign, including relational and substitutive views[12] (if x = 3, then x and 3 can be freely substituted for one another), is crucial for success in algebra;[3] a "limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra."[13] Much of the difficulty students have in their early algebra learning can be traced to a "compulsion to calculate;"[14] in other words, a purely operational view of the equal sign.

IMPLICATIONS: Developing a sophisticated understanding of the equal sign, including relational and substitutive views, is crucial for success in understanding and using algebra

Encouraging students to move beyond operational views of the equal sign when young can prevent difficulties with algebra later

"From a mathematical point of view, the equals sign is not a command to do something. Rather it is a signifier of a very important relationship – that of equality"

Darr, 2003[15]

"In the United States the equals sign is rarely defined and is often used interchangeably with computational terms such as 'makes'"

Jones et al, 2012[2]

Lucy Rycroft-Smith & Tabitha Gould, 2020

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