Example Research Summary Package

Authors

Representing the work of
Example Research Summary Package

Sections in this document

Research Summary: Early angle conception and measurement .................................................................3
Example activity types from the Swan Framework, as applied to student actions .................................12
Three waypoint content examples: exploratory waypoint, waypoint and landmark waypoint ............13
References..................................................................................................................................................17
Research Summary: Early angle conception and measurement

Author: Rachael Horsman

Research landscape

Angles combine ideas of both geometry and measurement. Multiple definitions exist, with angles being considered as (a) a geometric shape, (b) a measure, or (c) a dynamic rotation (Henderson and Taimina, 2005). It seems sensible, if not essential, to support learners in integrating both the static and the dynamic (described below) in order to understand angle measure (Clements & Battista, 1989, 1990; Mitchelmore & White, 2000).

In the work of Mitchelmore and White (2000, p.214-216) three stages of abstraction are identified:

• situated angle concepts where students identify, characterise and operate on angles;

• contextual angle concepts where students establish properties of angles and use these to solve problems; and

• abstract angle concepts where students formulate and use definitions, give informal arguments and hierarchies to previously discovered properties and finally develop deductive arguments.

This Research Summary concentrates on the first of these stages.

When exploring angles a variety of situations and contexts will be investigated. Initially the concept starts with children’s own dynamic perspective: enacting turns for themselves (Gates & Griffin, 1988), watches and wheels (Magina, 1994); alongside their intuitive use of parallel and perpendicular when they first start trying to build with blocks and similarly when they align tiles (Sarama and Clements, 2009). Completing tangram-type puzzles gives opportunities to compare the size and hence congruence of angles and angle measure. Drawing these activities and discussions together, bringing them into geometry lessons and making connections to the angle measure, enhances understanding (Sarama and Clements, 2009).

It is advised that three distinct settings are investigated, independently to begin with (Mitchelmore and White, 2000):

Rotations

It is possible to categorise rotations as either limited or unlimited about an interior point: e.g. a key rotating in a lock (limited), the hands of a clock (unlimited). Rotations can lack a clearly identified centre and rotating line hence the association to angle may not always be made by learners.
Angles (hinges)
Hinges are the prototypical angle. Mitchelmore (1998, p.269) categorises these into three distinct types:

- **I-Hinges** – a single, linear object hinged about one end rotating between well-defined limits; e.g. a door

- **V-Hinges** – two linear objects hinged about a common end point; e.g. a book cover

- **X-Hinges** – two linear objects hinged about a common interior point; e.g. scissors

Importantly, these can be viewed as static or dynamic (Magina, 1994, p.45) and present a good opportunity to invoke the principle of variation in terms of direction and orientation of an angle to be measured.
The definition of a dynamic angle considers it as a measure of quantity of rotation needed to bring one of its sides from its own position to that of the other side (Heath, 1956, p. 179 in Magina, 1994). Mitchelmore (1998) models this by considering a pair of scissors. The dynamic model would show the scissors opening; a static model the result of that opening. In many cases the two models are not in conflict. However, at times more careful thought is needed – such as with an open door. The dynamic model may consider the angle through which the door is opened whereas the more natural static one would be to measure the angle between the given position and the fully open position (resulting in two different angles). Hence, a correct, unambiguous and efficient labeling system is needed.
Bends (angular)

Bends are defined as two line segments with common endpoints. Moving forward along one line, a turn is needed to continue the journey along the second line: e.g. external angles of a polygon, angles of inclination, turtle geometry. In contrast, curves and arcs will be considered in more detail in a later research summary, although they may be compared and contrasted to angular bends within the waypoints discussed below.

Providing opportunities for students to explore each of these contexts is critical for the development of their understanding of geometry (Browning et al., 2007).

After their explorations of these three independent settings students are encouraged to become aware of their shared characteristics, collecting their experiences together. As Skemp (1986) describes, this abstracting (classifying according to these similarities) results in an abstraction: in this case the concept of angle.
Framework map for Early angle conception and measurement

Providing opportunities for students to explore each of these contexts is critical for the development of their understanding of geometry (Browning et al., 2007).

After their independent exploration students are encouraged to become aware of the shared characteristics, collecting their experiences together. As Skemp (1986) describes this abstracting (classifying according to these similarities) results in an abstraction: in this case the concept of angle.
Implications for the Framework

As in the other geometry themes, specifically those dealing with measurement, the initial waypoints in this saved search begin with an exploration of the concept. Pupils identify objects that rotate (dsm0.2) linking in with the Describing Movement theme. Pupils identify angles, in 2D and 3D contexts in both static and dynamic situations (ang0.3) and in instances where forward motion is combined with bends or slopes(ang0.4). Importantly in all these waypoints static and dynamic examples will be considered, including some examples of embodied maths such as following lines on a playground.

In each of these settings, angles are compared directly – superposing angles/representations of turn (ang0.1, ang0.3, ang0.4); in addition some attention is paid to representing an angle between two planes using two line in 2D (ang0.3). Clear representation and conventional labeling of angles is also discussed in these waypoints and each also connects to a waypoint concerning the conventions of labeling line segments (outside this saved search).

A unit of measurement is developed through the initial use of right angles as a unit (ang0.1, ang0.2, ang3.2). Then (as with other measuring themes) there follows the development of interim arbitrary units whilst pacing off angles until finally the unit of degree is introduced as the ‘standard’ unit (ang1.1). At the same time benchmarks are refined, such as 45˚, linking to the number waypoint Seeing Double (ast1.3) (outside this saved search). This follows the development of early measurement mirrored in the areas of length, area and volume. Further information can be found in the Research Summary Developing a sense of measure.

Each of the settings is revisited in order to identify, estimate and measure the angles (ang2.1, ang4.1, ang2.4). Finally the settings are drawn together (ang5.1) resulting in pupils identifying their similarities, developing a rich and secure understanding of the concept of an angle and its measurement.

This theme also overlaps significantly with the themes of: 2D Shapes; Loci and Construction; Describing Position; Describing Movement; Isometric Transformations; Congruence; and Symmetry.
References

Cited sources


Additional sources


Example activity types from the Swan Framework, as applied to student actions

<table>
<thead>
<tr>
<th>Student Action</th>
<th>“Sample classroom activities.” (Swan, n.d., p. 1)</th>
</tr>
</thead>
</table>
| **Performing** | • “Rehearsing well-defined procedures through exercises and études.”  
• Systematically using and memorising terms and notations.” |
| **Classifying** | • “Observing and manipulating mental objects.”  
• Identifying and describing attributes and sorting objects accordingly.  
• Creating and identifying examples and non-examples.  
• Creating and testing definitions.” |
| **Representing** | • “Interpreting a range of representations including diagrams, graphs, and formulae.”  
• Translating between representations and studying the co-variation between representations.” |
| **Analysing** | • “Studying and modifying mathematical situations.”  
• Exploring relationships between variables.  
• Comparing and making connections between mathematical structures.” |
| **Arguing** | • “Making and testing mathematical conjectures and procedures.”  
• Identifying examples that support or refute a conjecture.  
• Creating arguments that explain why conjectures and procedures may or may not be valid.” |
| **Estimating** | (this student action is not part of Swan’s original framework) |
| **Modeling** | • “Making suitable assumptions to simplify a situation.”  
• Representing a situation mathematically.  
• Identifying significant variables in situations.  
• Generating relationships between variables.  
• Identifying accessible questions that may be tackled within a situation.” |
| **Solving** | • “Planning an approach to a problem.”  
• Selecting and applying appropriate mathematical concepts and procedures.  
• Selecting and using mathematical tools, including technology.  
• Carrying out a plan, monitoring progress and changing direction, where necessary.  
• Making generalisations based on the results.” |
| **Conference participants** | • “Interpreting, adopting and continuing a strategy.”  
• Comparing alternative strategies, identifying relative strengths, weaknesses and domains of application.  
• Reflecting on solutions: examining for reasonableness within the context.  
• Reflecting on strategies and arguments: where might they have been improved?  
• Interpreting and testing mathematical models: Are they adequate? What are their limits?  
• Making connections with previously encountered problems.” |
Three waypoint content examples: exploratory waypoint, waypoint and landmark waypoint

Figure 1: Inset map showing the waypoints whose details are listed below

Key:  
- **Exploratory waypoint**
- **Landmark waypoint**
- **Waypoint**
- **Development of concept, skill, procedure**
- **Use of concept, skill, procedure**
Example waypoint details

(glossary links under development; not shown)

<table>
<thead>
<tr>
<th>ang0.2 Finding right angles</th>
<th>Student Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>Exploratory waypoint.</td>
</tr>
<tr>
<td>What</td>
<td>Identify right angles, and multiples of, in 2D and 3D using practical equipment.</td>
</tr>
<tr>
<td>Why</td>
<td>To develop benchmarks for angle measurement; to recognise the importance and high frequency of occurrence of right angles and their multiples; to begin to build understanding of the properties used to define or classify shapes.</td>
</tr>
</tbody>
</table>

**Performing**
- Identify right angles in objects and images.

**Classifying**
- Sort objects, images and geometric patterns into groups of those with and without right angles.

**Representing**
- Represent right angles in a variety of ways, for example: using finger and thumb; corner cards; traced angles; and frames, including L’s, T’s and X’s.

**Analysing**
- Explore how right angles fit together and the relationship to a straight line and around a point.

**Leads out to**
- ang0.3 by theme Angles
- 2dc1.1 by theme Angles
- 3dc1.3 by theme Angles
- 2dc1.1 by theme Disembedding and embedding
- 2ds2.1 by theme 2D
- 3ds4.1 by theme 3D

**Research**
- This node is linked to

**Glossary**
- This node references 2D, 3D, angle, corner, geometrical, line, measure, multiple, pattern, point, property, right angle, shape and straight.
### ang0.3 Identifying and comparing angles

<table>
<thead>
<tr>
<th>Type</th>
<th>Performing</th>
<th>Classifying</th>
<th>Representing</th>
<th>Analysing</th>
<th>Modelling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>• Identify angles (static and dynamic) as a measurement between two lines.</td>
<td>• Classify angles as bigger or smaller than one, two or three right angles.</td>
<td>• Represent physical angles using, for example: in 2D two strips of card and a split pin or a bendy straw; in 3D a folded paper wedge; so that these can then be reproduced on paper.</td>
<td>• Compare the size of given angles, for example: by eye; by physically overlapping them; or by using other equipment.</td>
<td>• Model, explore and explain ideas about angles using physical models: for example, using two strips of card and a split pin; a folded paper wedge; or a bendy straw.</td>
</tr>
<tr>
<td></td>
<td>• Identify angles (static and dynamic) as a measurement between two planes.</td>
<td>• Order a selection of angles by size.</td>
<td>• Use an arc to label an angle in a diagram.</td>
<td>• Consider the (non)-implication of the size of the arms of an angle or arc marking an angle.</td>
<td>Model an angle between two planes as the angle between two lines in 2D.</td>
</tr>
</tbody>
</table>

**What**

Identify and compare angles in 2D and 3D. Compare angles: predict, justify and confirm practically whether one is larger, smaller or equal to another.

**Why**

To build the concept of an angle as a measurement between two lines that meet (or cross) at a point or two planes that meet along a line.

To introduce the beginning of number sentences, through the informal use of $<$, $>$, and $=$.

**Glossary**

This node references 2D, angle, arc, bend, corner, diagram, difference, line, measure, model, plane, point, right angle and shape.

**Leads out to**

- lea0.1 by theme Comparing
- ang3.2 by theme Angles
- ang4.1 by theme Angles
- ang1.1 by theme Angles
- mea0.5 by theme Angles

**Research**

This node is linked to:

### Comparing measures

<table>
<thead>
<tr>
<th>Type</th>
<th>Student Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Performing</td>
<td>- Directly compare the angle, length, area or volume of objects and shapes (where appropriate).</td>
</tr>
<tr>
<td></td>
<td>- Recognise and identify lengths, areas, volumes and angles that can be directly compared.</td>
</tr>
<tr>
<td></td>
<td>- Use comparative language, such as: taller, shorter, wider, narrower, longer, shorter, larger, smaller.</td>
</tr>
<tr>
<td>Classifying</td>
<td>- Identify objects/shapes with a larger or smaller measure (angle, length, area or volume).</td>
</tr>
<tr>
<td>Representing</td>
<td>- Understand, use and justify the symbols: &gt;, &lt;, and = between objects/shapes to represent comparisons.</td>
</tr>
<tr>
<td>Analysing</td>
<td>- Compare and make connections between the different methods of direct comparison of angle, length, area and volume.</td>
</tr>
<tr>
<td>Solving</td>
<td>- Use the logic of if ( a &lt; b ) and ( b &lt; c ) then ( a &lt; c ) in comparing angle, length, area and volume.</td>
</tr>
</tbody>
</table>

**What**
- Recognise how objects and shapes can be compared in a mathematical sense, specifically in terms of angle, length, area, volume and weight (referred to as mass once units have been introduced).
- Identify measurable attributes of shapes and objects; 2D shapes have a perimeter (a length), angles and enclosed area, 3D shapes have edges (length), faces (areas), angles (on the faces) and an enclosed volume. They also have a weight.
- Recognise the similarities and differences in the actions carried out in order to directly compare in each measure.
- Recognise how some actions, including decomposition, do not change the overall angle, length, area, volume or weight.

**Why**
- To develop the understanding of length, area, volume and weight as comparable attributes in objects and shapes;
- To develop the mental ability to compare objects;
- To begin to understand which transformations can be performed such that the comparable attribute remains unchanged.

**Leads out to**
- mwd0.2 by theme Mass, weight and density
- lea0.6 by theme Relationships
- ang1.1 by theme Comparing
- are1.1 by theme Comparing
- vol0.9 by theme Comparing
- vol1.3 by theme Comparing
- ppl1.1 by theme Comparing
- mea0.9 Measuring

**Research**
- This node is linked to

**Glossary**
- This node references 2D, 3D, angle, area, attribute, decomposition, difference, edge, face, length, mass, measure, perimeter, shape, symbol, unit, volume and weight.
References

