CAMBRIDGE Mathematics

In More Detail:
Supplement to March 2018 Framework Update

Contents

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Perspectives and beliefs

What guiding questions and principles inform the design of the framework?

If we hold that improving the coherence of students' mathematical experiences can contribute to improving the way they learn with understanding, then our design should address the following questions...

1. How can the contents of a maths curriculum framework be expressed in a way that:
    - has core features that designers, teachers, and subject experts can interpret and assess relative to their context?
    - emphasises connections?
    - tracks and describes research influences in localised parts of the framework and across the structure of the framework?
2. How should this goal structure our design processes and the evaluation of outcomes?

What is coherence and how does it affect your approach?

'Coherence' is frequently called for across the curriculum design and mathematics education literature as a way to increase effectiveness of teaching and learning. Broadly speaking, it describes a whole formed by component parts working productively together. In this sense its meaning seems (appropriately) coherent, but which whole, which parts, and what constitutes working together vary according to the context.

The word is used to convey a similar meaning about culture and content at multiple scales, from beyond the mathematics curriculum to focusing deep within it. The number of people involved in generating coherence at different scales ranges from everyone to no one. It could be generated through interactions between groups in the wider dominant cultural setting, or alignment of communities and practices in an education system, or alignment of individuals with school culture. It could be generated through logical sequencing of school content in a domain by curriculum designers and teachers, all the way down to individual moments of learning and, finally, to the disembodied structure of mathematics learning and the domain of mathematics itself.

The implications of coherence both for equity and for the structure of the curriculum may be very different according to the assumptions made when invoking the concept. In its broader applications, coherence is taken to be a property of a 'common culture' of which individuals in a school setting are a part. This common culture provides "a framework of common understandings within which society and individuals within it might function coherently" (Pring, 2012, p. 49). However, within the larger cultural context, there are subcultures (Hall, Morley, & Chen, 2005) and communities of practice (Lave & Wenger, 1991) with particular (and particularly influential) elements of coherence in their local contexts. These may sometimes be at odds with other groups or with the larger culture.

Depending on whether coherence is viewed more as arising from a unified perspective or from coordination of diverse perspectives, solutions for improving it tend to fall into two categories: finding an intersection of approaches and applying it across the system (Schmidt, Wang, & McKnight, 2005), or finding points at which members or products of different groups can interact and enhancing those interactions (Hall et al., 2005; Robutti et al., 2016; Thurston, 1990). These approaches need not be mutually exclusive, but can be invoked depending on the scale of the suggested intervention (Schmidt et al., 2005).

In the context of individual thought and systems of ideas in mathematics, Cobb (1988) describes coherence at the scale of knowledge construction in an individual's development of understanding. This occurs at the moment when they integrate pieces of a solution learned from a teacher into a single coherent solution. The word can also be used in a way that is completely disembodied, to describe standards representing the learning of mathematics (Schmidt et al., 2005), or the nature of mathematics, (Thurston, 1990), logic, or reasoning (Dewey, 1938). Even when used to describe qualities of ideas rather than the actions of the people who hold them, there is not a domain-wide shared basis for agreement on many of the specific sequences or structures that could contribute to coherence in and of themselves

We apply these two perspectives on coherence, the cognitive and the cultural, to our design in different ways. From a cognitive perspective, we represent mathematics learning according to a consistently applied set of considerations for *what* is described and *why*, and *how* it can be experienced by students through their actions (see How is mathematics learning expressed at a waypoint?). We share this representation with the communities of practice that generated the research, who review and improve it.

In order for the framework to contribute to coherence in the cultural sense, we incorporate shared model-building into our design process so that the construct may help to foster shared meanings and practices in communities with diverse perspectives. We generally can't control how widely across a system the framework would be used, but we can seek to make it appropriate for use in coordinating curriculum approaches, materials, and enactment. We first formed a writing team whose members collectively belong to each of the three types of professional communities we have identified as potential users of the framework – curriculum designers, resource designers, and teachers. Throughout the design process we have consulted with experienced representatives of the curriculum designers, resource designers, and teachers communities to sense-check the meaning, validity, and usefulness of the framework in progress. We are entering a stage of design where we are beginning to conduct an expanded and more formalised version of this evaluation.

What is your perspective on connected understanding and how does it affect your approach?

Our perspective on understanding in mathematics learning and the nature of the connections are key to the design of the Framework. Here we summarise these topics which have been extensively reviewed in the sources we cite below, among others.

Mathematicians, activity designers, and mathematics education researchers each have perspectives on the development of understanding in mathematics that we have found helpful. A broad functional distinction between understanding or not understanding some mathematics is held roughly in common. If someone does not understand some mathematics, even if they "know" it as a fact or a process, they will not be able to use it flexibly or powerfully (Usiskin, 2015). The NCTM Principles and Standards for School Mathematics framework distinguishes between conceptual understanding, factual knowledge, and procedural fluency as three interdependent components of *proficiency*, the ability to use mathematics well (Mathematics, 2000). Hiebert & Carpenter (1992) note that understanding allows students to be inventive and intuitive with mathematics, to remember it more effectively and efficiently, to transfer what they know to new contexts, and to build positive and productive beliefs about the domain.

If this is what understanding achieves, how does it develop, and how do students use it to build the structure of mathematics within themselves? Thurston (1990) describes the structure of mathematics as tall, broad, and connected. It is 'tall' since concepts tend to build on other concepts; the structure of this is "like a scaffolding, with many interconnected supports" (p. 2). It is broad because it involves many interconnected concepts, and this breadth is necessary to support the height of the tall structure – without a broad enough base, individuals' understanding would have no support for growing 'taller' past a certain point. This connected structure makes it "compressible," allowing students and mathematicians to effectively access and use the web of conceptual understanding and practices they have built in size and complexity over time (Tall, 2013; Thurston, 1990). This understanding and compression facilitate transfer, so that students can apply their understanding creatively in appropriate contexts, even if those contexts are novel (Bransford, Brown, & Cocking, 2000).

All of this suggests that the development and growth of understanding is made possible by a web-like scaffolding structure that each student builds for themselves, with classroom experiences playing a large part. In deciding what this means for the design of the framework, we draw on existing research for ideas of both the connections made within the learning mind and connections within the domain of mathematics as a logical system, while remaining aware of the limitations of what may be currently "known" about either for any given area of mathematics. There is a growing body of empirical work on learning progressions, trajectories, or pathways, concerning the connections that students make in building conceptual understanding (Black, Wilson, & Yao, 2011; Daro, Mosher, & Corcoran, 2011; Maloney, Confrey, & Nguyen, 2014). Michener (1978) developed a framework for mapping mathematics understanding that expresses concepts in directed graphs like ours; while it was developed rigorously it has so far been actively applied only to limited areas of the domain. These studies provide invaluable insight but so far tend to be concentrated in certain areas of the mathematics education literature (concepts commonly introduced in pre-primary and primary years), and the process of producing and testing conjectures is intensive and slow-moving relative to the time frame for this project. While this research influences our work where it exists, there is not enough of it to form the basis of our design.

When we take a step back from the idea of directly mapping the paths that different proportions of students will take, we can look at other ways connections are described within the domain. Hiebert & Carpenter (1992) describe students forming "rich networks of knowledge" (p. 74) to which elements are added *from the time they are first experienced* rather than after students are proficient. Students can then benefit from considering those ideas, together with related concepts or processes that offer experience of the concepts in action (Denvir and Brown, 1986, Skemp, 1979).

Although we cannot map these rich networks literally, we can still focus on identifying elements of the domain that students experience and ways that relationships between those elements are characterised in existing research. In this way we can avoid making claims in the absence of data about which pathways students are more likely to take, but we can provide support to designers and teachers who may have their own working hypotheses or guidelines. By linking existing research on learning trajectories to such a structure, we can also show where more research on learning trajectories might make a difference to decision-making in design and teaching.

We have elected to build a connected web of relationships between elements of mathematics that students experience, which we call waypoints. We describe these in terms of actions students would take in order to have some facet of this experience that theory suggests should contribute to building understanding (Swan, 2014). Unlike an ordered set of standards or learning objectives, for most waypoints we do not describe what indicates that a student has "finished" with that content. We expect that the path a particular student takes through this web would involve visiting elements more than once, backtracking and circling around as they build their understanding from prior experience and the neighbourhood of experiences of related concepts.

This focus on student experiences of domain content also provides common ground for shared understanding and communication between designer, teacher, and researcher roles, both facilitating individual work and helping to coordinate it with the work of others. In many jurisdictions, communication about connections that are important in different contexts is infrequent in education settings. This is partly because there is little to no overlap between professional communities involved in various stages of mathematics education and practice (high school, undergraduate, mathematical research, applied mathematics, computer science, statistics, etc.) (Thurston, 1990) and there are important differences between the ways these communities consider mathematics (Usiskin, 2015).

How is your approach informed by theory?

Our focus is on how our design is informed by existing theory rather than on how it might contribute to developing theory. In this way our use of theory has been predominantly consistent with a *design as intention* orientation (Collins, Joseph, & Bielaczyc, 2004), that is, making the initial design both internally coherent and consistent with well-defined theories informing it, in addition to the professional experience of the designers (Ruthven, Laborde, Leach, & Tiberghien, 2009).

Different aspects of the design require us to coordinate informed assumptions about:

- the nature of mathematics learning, including what perspectives have been identified as essential and in need of more support, i.e. what our model might highlight,
- the nature of professional knowledge of mathematics learning and how it might be coordinated across multiple professions, including what particular goals for coherence our design might contribute to achieving , and
- the nature of design and how we should carry out our work in order to achieve our goals for the quality and usability of the framework.

Each of these considerations draws on distinct sets of theories which apply in particular ways to our design, depending on the scale of the particular design questions being addressed. This is one way in which design projects in education may differ from traditional research projects; research may make its clearest theoretical contributions from a more narrowly focused theoretical base, while design outcomes depend on preparation for complex contexts in the real world, and what Sfard calls “theoretical pluralism” (2003, p. 355) can be beneficial (Kieran, Doorman, & Ohtani, 2015). Multiple theoretical perspectives should be brought to bear on a successful design, as most theories are only constructed to look at a particular facet of situations a design may face, and many theoretical perspectives are either complementary or else talking at cross purposes rather than truly contradicting each other (Sfard, 2003).

Boundaries and constraints are still necessary to make any model simple enough to be coherent. Theoretical frames must be meaningfully coordinated, and we need to track how we relate a theory to a design choice – whether it has been “illuminating, inspring, guiding, systematizing, or even constraining” for us (Kieran, Doorman, & Ohtani, 2015, p. 30).

One useful way to characterise theoretical frames in design projects is according to their scope of application. Kieran et al. (2015) and Ruthven et al. (2009) discuss the role of grand, intermediate, and domain-specific frames. Grand-frame theories (e.g. constructivism) shape fundamental perspectives on learning but are not directly applicable to the specifics of design. Intermediate frames have a narrower focus that nevertheless can be applied across areas of mathematics learning. Their origins may be “primarily theoretical or…based to a large extent on deep craft knowledge” (Kieran et al., 2015). They tend to have implications that can be more clearly and directly applied to design decisions given our *design-with-intention* orientation.

Domain-specific frames are even more narrowly centred on particular concepts and processes (Kieran et al., 2015).

Some perspectives have been particularly influential on our work so far; the grand frame perspectives are of course held in common with many similar efforts while some intermediate and domain-specific theories inform our approach more specifically. In *What is your perspective on connected understanding* (above), and *How is mathematics learning expressed at a waypoint?* (below), we discuss a set of intermediate frames that has led to us expressing mathematics learning through student experiences and actions. In *What is the basis for your design approach* (below) we discuss intermediate-level theories informing our perspectives on design relative to our goals for the framework. Our intermediate-level perspectives on professional knowledge and knowledge-sharing include models of pedagogical content knowledge from the perspective of designers and teachers (Shulman & Shulman, 2004), models of co-design practices for knowledge representation (A. Blackwell & Green, 2003; Sanders, 2002; K. Schmidt & Bannon, 1992), and theories of boundary objects (Star & Griesemer, 1989) and boundary negotiating artefacts (Lee, 2005); we are currently working on detailing this in a paper for publication and will add a summary to this FAQ. Domain-specific theories are discussed in research summaries corresponding to the content and structure of small related sets of waypoints in the framework.

Methodology

What is the basis for your design approach?

We consider our approach to be research-informed design. It is a qualitative, interpretive process of expressing mathematics learning, combining theory and empirical research from a variety of sources, with descriptions and experience from practice in a way that is clearly explained and documented at a fine-grained level.

We want research to influence the design and content of the framework as much as possible, in a way that is meaningful and valid. The degree to which this *is* possible depends on our priorities and constraints for the design, what is in the existing literature, and our capacity to find and review relevant work. As mathematics curriculum framework designers have noted in similar contexts, there is more to draw on in the literature for some areas of mathematics education than others, and it is also more feasible to employ review methods that identify relevant, essential areas and themes than to complete exhaustive systematic reviews of work in every subdomain (Cunningham, 2017; Ferrini-Mundy & Martin, 2003; Sfard, 2003). This means that while a design can be grounded in research it cannot be *prescribed* by research, and we do not suggest that our design is "what the research says" to do. Rather, we want our design to draw on existing research in the context of our goals for the framework as a whole. These include considerations that are as much about how content is portrayed, recognised, and put into action by different users as they are about the nature of concepts, skills, practices etc. that have been characterised in mathematics learning.

We draw on models for design processes in education that have been developed and refined within design research methodology in education for over twenty years (Cobb, Confrey, Lehrer, Schauble, & others, 2003; McKenney & Reeves, 2012). Particular aspects of design research that make some of its methods appropriate for us (apart from a general focus on design) include: linking specific design priorities and choices to theory; using initial design work to develop design principles that inform ongoing work; going through iterative cycles of design in which feedback on work in progress is incorporated into new design versions and practices; and participation in design by experts in multiple relevant communities (Barab & Squire, 2004; McKenney & Reeves, 2012; van den Akker, Gravemeijer, McKenney, & Nieveen, 2006).

Our aims differ in some respects from those of those of design researchers. For example, design methods are typically employed in primary research in education with the goal of producing or refining theories of learning. This is done on the basis of new data generated from direct implementation of research-based designs in classrooms or education systems (Barab & Squire, 2004). These designs and implementations are engineered so that together they may coherently contribute to theories of learning more broadly. In contrast, we rely on reviews of existing research to inform initial design choices, which we adapt and refine using feedback from expert evaluation. We aim to conduct and document our work in such a way that our design might later be able to contribute to research. However, for many areas of the framework the closest we will come to generating data from the design before it is released will be through expert interviews and group surveys for face validation of the content and the structure of the Framework. In that sense, our methods are those that would be used in the beginning stages of a design research project.

Similar approaches have been described by framework development projects in other contexts. In a retrospective review of the standards writing process for the NCTM *Principles and Standards of School Mathematics* framework, the writers noted that a set of theoretical perspectives emerged as important influences over time and described their design work as “researchlike” because it also involved collecting, analysing, and incorporating feedback on work in progress (Ferrini-Mundy & Martin, 2003). Of work currently in progress, the UNESCO Institute for Statistics is currently developing the Reference List & Coding Scheme (RL&CS) framework, intended to provide rich qualitative support for mapping theory and curricula to assessment frameworks. This project’s interpretive approach similarly required the designers to evaluate trustworthiness on the basis of practical value of the construct, as expressed through feedback on work in progress by expert members of the communities that would be making use of it (Cunningham, 2017).

What is your design process?

We treat construct design and content writing as intertwined processes. We laid the groundwork for the design with high-level review of theories and approaches and we continue with cycles of review, writing and refinement.

Laying the groundwork

Initially, we reviewed a variety of perspectives on classification schemes and “big ideas” in mathematics education, as well as curriculum frameworks and content documents from a selection of jurisdictions. We used ideas from this process to create a tentative “top-down” way of dividing parts of the domain among members of the writing team. At the same time, we imagined what we might need from the construct from the bottom up and reviewed existing frameworks for conceptual understanding in mathematics (Freudenthal, 1983; Michener, 1978; Pirie & Kieren, 1994; Schoenfeld, 1992; Skemp, 1979; Tall, 1988, 1999; Usiskin, 2015; Vergnaud, 1996 among others) and for learning with understanding (Bransford et al., 2000; Hiebert & Carpenter, 1992; Kieran, Doorman, & Ohtani, 2015; Martin A. Simon, Nicora Placa, & Arnon Avitzur, 2016; Sfard, 2003; Simon & Tzur, 2004; Swan, 2014, among others).

Content writing and construct design

Our focus has tended to move in the direction of “traditionally early” to “traditionally later” topics, although we are careful not to reproduce the order of literature review topics as a necessary “order” of topics in the framework. Each writer, having adopted a major domain area, referred to the initial literature review to examine likely subdomain areas and began an initial review of a topic. Through discussion within the writing team, discussion with mathematics curriculum researchers and reference to the initial review of frameworks, we began writing about elements of these topics according to a task design framework proposed by Malcolm Swan (2014) for learning with understanding, on the basis of its consistency with conceptual and teaching-learning frameworks developed by others and its demonstrated relatability to designers and teachers. While proposing this structure, we also recorded information from reviews that did seem important but did not necessarily “fit” and we continue to examine the implications of this structural choice. Many of these have subsequently been addressed by adaptations to the original framework, which we will discuss in more detail in future papers.

Currently we are in a cycle of writing, discussion, feedback, and refinement of the content and the construct. We have developed a set of tools for writing new content into the structure of the framework, searching and visualising existing content, and collecting reviews of content. Each writer works according to a cycle of (1) literature review, (2) generation of waypoints, relationships, glossary definitions, research records and any other features called for in a particular area, (3) internal discussion and review, (4) external review, and (5) response and refinement. We are about to implement a semi-structured protocol for external review, in which we will provide reviewers with access to the visualisation and search tools used by the writing team. Currently, external review plays the same role but was conducted more informally, with each writer documenting feedback discussions in their own way. This external review process is the basis for initial face validation of the content and structure of the framework. This will be followed by further face validation of the construct through structured group methods for identifying areas of consensus and dispute among experts (e.g. Delphi panel method, Clayton, 1997), and we are approaching the point where we feel there will be enough to provide for discussion using these methods.

At the same time, we are working to characterise relevant ways of working among potential users of the framework so that we can anticipate discrepancies between our initial design assumptions and what might be necessary in order for us to meet our goals for the design of the framework and user interfaces. In addition to user surveys, we considering methods for evaluating representations and interfaces that we would use when we are closer to working directly with potential users of the framework.

How does the literature inform the design and content of the framework?

Areas of the framework are designed and written on the basis of literature review and consultation with researchers and subject experts with broad acceptance in the mathematics education community. Our capacity for reviewing the vast amount of literature applicable to mathematics learning from age 3-19 is constrained, and a systematic literature review in every area of school mathematics is not possible for us as a prerequisite for design. However, many projects have undertaken portions of such a review and we refer to their work whenever possible. For the sake of identifying important themes and findings across the breadth of the framework we are following a semi-structured review process that includes both keyword database search, purposive sampling according to syntheses, meta-analyses, recommendations from consultation, and breadcrumb search starting from widely accepted texts such as recent research handbooks (Thomas & Harden, 2008). Similar to the process for thematic synthesis that Thomas and Harden describe, we make greater use of grey literature than a structured review with a more focused aim would, but we feel this is appropriate to our aim of broadening initial coverage of the literature.

Our inclusion criteria for this process are broad and functional, with the understanding that inclusion and use of sources will be subject to further review as part of our external expert review process (described below). We note and exclude any source from a particular review that we judge to be irrelevant or lacking reasonable support for its claims. To facilitate expert review and our goal of transparency, we record metadata for research sources to help characterise and communicate our influences. This includes…

- source level of influence (main, secondary, or excluded
- found by search method (recent synthesis or handbook, expert recommendation, database search, search engine, breadcrumb search)
- publication context and intended audience
- broad category of focus (meta-analysis, other review, mostly empirical, mostly theoretical, mostly philosophical, mostly individual)

When sources are used in writing framework content, they are entered into the framework database and linked to that content. This makes it possible for writers, reviewers, and users to summarise and examine the influences that have contributed to specific areas of the framework.

How did you develop the structure of the framework (ontology)?

Our design process has followed the general cycles of ontology development described in Fürst, Leclère, & Trichet (2003, p. 80). We started with an informal conception of the mathematics learning and relationships we might want to model in a few foundational areas of the framework, like *Counting*. We built an initial conceptual model and revised it many times based on discussions between writers working in different areas. We started to agree on more structural features and ways of characterising content that could remain consistently meaningful when applied to different areas; this became a rough ontology, which was then subject to further revision as new issues emerged that needed to be resolved. In later stages of this process, we formalised the ontology into the framework database, moving from paper and spreadsheets to a consistently applied custom format; however, this format is still being actively revised and updated as issues arise and consensus begins to form around existing questions. The content of the framework is now stored as graph data using a graph database platform, Neo4j, and accessed using an interface designed around our ontology. Elements of content, the entities of the ontology, are *nodes* and relationships between them are *edges* in the graph.

An ideal ontology would define entities and relationships in such a way that there was no ambiguity or overlap, with a minimum of real-world messiness left out. When there is not enough information available to do this, an ontology may remain in an intermediate state of development, with a more formal part consisting of clear or at least consensual structures and definitions, and a more informal part consisting of elements “in development” whose structure or meaning may not yet be agreed. Once a version of a formal representation exists, designers can build interfaces, groupings, and hierarchies that make it meaningful and accessible as a shared representation, which then fuels further cycles of development (Fürst et al., 2003).

How do you write and revise content for the framework?

Framework writers review sources in areas of the literature appropriate to targeted areas of the framework. They make choices informed by the ontology about what to express separately or in combination at waypoints, and which relationships to other waypoints are important to include. This is done as consistently as possible and reliability between writers, which is informally checked in writing discussions, has steadily improved. They note apparent consensus or possible lack of agreement in the context of the theoretical stance and questions driving each study. The specific set of choices and influences for each area that becomes synthesised into a set of nodes and edges is documented in Research Summaries and the structure, content and research base in that area is then subject to evaluation and revision centred on that research summary. Connections between smaller areas of the framework are made during this process and also through discussion with other writers in a process of internal review. Over the course of content development, writers can use search and visualisation tools to consider implications of their work and fuel discussion. Important terms are linked to the glossary where they may be defined according to additional literature review if they are not already there.

We use a set of web-based tools to visualise, create, and edit text and structural content in the framework database, and to collect and review comments on particular features. The development of these tools has evolved alongside the structure of the framework itself according to the needs of writers and reviewers. We anticipate that our experiences designing and working with these tools will help inform the interfaces eventually used by others to access framework content.

How do you evaluate whether the structure and content are valid representations of mathematics learning?

Reflexivity – the writing team and collaborators

We record our own perspectives, beliefs, and experiences, and those of our collaborators, which contribute to particular approaches and decisions about structure and content. These considerations are a frequent part of our internal discussions and external collaborations. This is one method of confirmability discussed in Lincoln & Guba (1985)and employed in other research on curriculum frameworks (Cunningham, 2017; Ferrini-Mundy & Martin, 2003) and conceptual representation in mathematics learning (Confrey & Lachance, 2000).

Design narrative

We maintain notes from literature reviews and meetings, recording major design decisions about the structure and the tools we use to work with it. In addition, research sources are annotated and decisions regarding related sets of waypoints in the framework are explained in Research Summaries describing those waypoints. This serves some of the function of an audit trail, which Lincoln & Guba (1985) describe as a set of records adequate for making the research process transparent to aid the interpretation of outcomes. In design research, this includes being able to describe how and why the thing being designed came to be the way it is, and perhaps how it might best be developed further (McKenney & Reeves, 2012).

Face validation of mathematics learning content

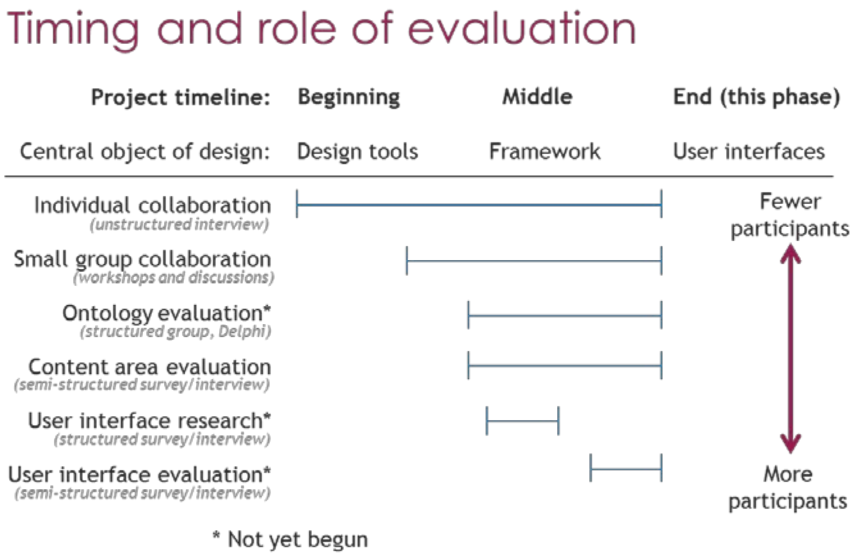
Wenger et al. (2011) suggests how an evaluation might look for value in the right place depending on how far along a design project is, from the point where the design is hypothetical, through when it starts to produce artefacts, through when those artefacts can be implemented and their value directly experienced.

In each stage, peer debriefing (review by external researchers), feedback from community members, and formal indicators of value are expected to provide a complementary perspective. In our case, we are implementing an external semi-structured questionnaire and interview protocol as a check on the value of content among educational researchers in specific content areas (Creswell & Miller, 2000), and we maintain informal engagement with representatives of our project's user/stakeholder groups. These are among the methods for establishing credibility and confirmability described by Lincoln & Guba (1985) that have also been used in recent curriculum framework development (Cunningham, 2017; Ferrini-Mundy & Martin, 2003). While the shortcomings of member-checking may include a lack of diversity of perspectives of those doing the checking or misinterpretation due to lack of familiarity with the way information is represented (Creswell, 2012), these methods may allow us to detect and work with any issues that could otherwise be problematic later on.

Face validation of the representation of content and its usefulness

Any decisions made about the ontology will affect how mathematics learning is expressed. In order to make content and links meaningful, we need criteria for deciding how "important" an idea or link between ideas should be before it can be expressed in the framework (otherwise everything would be connected to everything else and would not provide a meaningful or useful focus). This is necessarily subjective, so we need to test the validity of this method. We are in the process of designing a protocol for early face validation of the construct of the framework relative to our design goals. We have planned expert panel methods involving structured group interactions to identify widespread agreement or disagreement about elements of framework design (Bowker & Star, 1999; Clayton, 1997; Hartman, 2016). Further validation could be proposed in the future through experimental means once the framework can be linked to curriculum or resource designs implemented in schools.

Figure 1: A description of how and when feedback is positioned in our design process



The Structure of the Framework

How is mathematics learning expressed at a waypoint?

We define a *waypoint* as 'a place where learners acquire knowledge, familiarity or expertise'. The specification of waypoints in our ontology is based on characterisation of learning sequences by Michener (1978) and Swan (2014, 2015). Each waypoint contains a summary of the mathematics (the 'what') and why it is included (the 'why'). Those waypoints at the beginning of a theme, as described above, are additionally designated 'exploratory'. We recognise also that it is useful to bring different ideas together where the whole is greater than the sum of the parts - we identify these as 'landmark' waypoints.

Each waypoint is further characterised by what we call *student actions*. Student actions are derived from a framework for the design of tasks which support building conceptual understanding in mathematics (Swan, 2014, 2015). At a waypoint, specific examples of actions related to the *what* and the *why* of content at that waypoint show *how* students could engage with it.

Student Action	
Performance	Memorise and rehearse
Classification	Sort, classify, define and deduce
Representation	Describe, interpret and translate
Analysis	Explore structure, variation, connections
Argument	Test, justify and prove conjectures
Estimate	Make reasonable approximations or predictions
Model	Formulate models and problems
Solution	Employ strategies to solve a problem
Critique	Interpret and evaluate solutions and strategies

Adapted from Swan, M. (2014) and (Swan, 2015).

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